Noise-Induced Phase Bistability via Stochastic Rocking

Kestutis Staliunas,[1,2] Germán J. de Valcárcel,[3] Javier M. Buldú,[4] and Jordi Garcia-Ojalvo[1]
[1]Departament de Física i Enginyeria Nuclear, Universitat Politècnica de Catalunya, Colom 11, 08222 Terrassa, Spain
[2]Institució Catalana de Recerca i Estudis Avançats (ICREA), Colom 11, E-08222 Terrassa, Spain
[3]Departament d'Òptica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain
[4]Complex Systems Group, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles, Madrid, Spain
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We study the effect of a randomly modulated harmonic driving on the phase behavior of a nonlinear oscillator. A multiple-scale analysis shows that the system is formally equivalent to a rocked oscillator, in which a modulated harmonic driving locks the system at one of two phases, both of which are in quadrature with that of the driving. This theoretically predicted noise-induced bistable phase locking is reproduced with numerical simulations of a stochastic Stuart-Landau model, and verified experimentally in a nonlinear electronic circuit.

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Introduction.—The interaction of noise with nonlinearities is known to lead in a counterintuitive way to the emergence of order in dynamical systems [1]. A particular example is that of noise-induced transitions [2], which in many cases result in bistable regimes created by fluctuations. Such noise-induced bistability has been reported in many systems, including lasers [3], superfluid helium [4], surface waves [5], and enzyme dynamics [6]. In those situations, for a large enough noise intensity the stationary probability density of the relevant variable changes from monomodal to bimodal, indicating the appearance of new, coexisting stationary states induced by noise.

Multistable steady states have also been found to be enhanced by noise in coupled nonlinear *oscillators* [7]. However, in most situations in nature nonlinear oscillators are not quenched but free running, and therefore the interest lies not in the phase becoming fixed at a particular steady value, but on it becoming locked with a reference signal. Since the phase of a nonlinear oscillator is marginally stable, it is not at all clear that constructive phase dynamics in the form of locking can arise from noise. Here we propose a mechanism that does induce bistable phase locking by noise.

A nonlinear oscillator can be phase locked by applying to it a coherent resonant modulation with sufficient amplitude. Such an injection locks the phase of the oscillator $\phi(t)$ to a single value, i.e., that of the input $\phi_{\rm in}(t)$, or at least limits the phase shift to a certain range ($|\phi(i) - \phi_{\rm in}(t)| < C$). Recently, however, a driving protocol has been proposed that leads to *bistable* phase locking. The method, termed rocking, consists of modulating the harmonic driving while being applied resonantly to the nonlinear oscillator [8]. Such modulation amounts to, basically, changing the phase of the driving back and forth between two opposite values, so that the oscillator cannot stably lock to any of the two switching phases, and ends up instead locking its phase to a value that is $\pm\pi/2$ shifted with respect to the input. Rocking has been reported experimentally in lasers [9] and in electronic circuits [10] for the case of a deterministic driving; here we consider the case of a stochastic driving.

Many oscillators in nature are affected by both periodic driving and fluctuations. An example is the circadian clock of mammals, which is subject to fluctuation-prone periodic driving by the light-dark illumination cycle. Bistable phase locking in this system can be related to the experimentally observed switching between diurnal and nocturnal regimes of activity within the same animal, depending on environmental conditions [11]. The results presented here might represent a minimal mechanism that could explain the above-mentioned switchings in a noisy environment. Another natural situation where spontaneous switching occurs is in the phenomenon of bistable perception in neurodynamics [12]. Given the importance of phase dynamics in the brain, and the ubiquity of noise in brain tissue, one could expect that effects similar to those described in this Letter might be relevant in that context as well. From a physical perspective, this phenomenon is closely related with the emergence of random-field induced order in two-component Bose-Einstein condensates [13]. In a quantum optics context, the type of fluctuations considered here could be provided by a squeezed noise [14,15].

Theoretical analysis.—In order to establish that noise can also have a constructive influence in the phase dynamics of nonlinear oscillators, we show in what follows that the modulation of the resonant harmonic driving characteristic of the rocking effect does not need to be periodic, but it can be *random*. To that end, we consider a minimal model of a limit-cycle oscillator, namely, the Stuart-Landau model, which corresponds to the normal form of a supercritical Hopf bifurcation:

$$\frac{dA}{dt} = (1 + i\nu)A - (1 + i\beta)|A|^2 A + \eta(t). \qquad (1)$$

$A(t)$ is the slowly varying amplitude of the oscillator, whose state is given by $x(t) = A\exp(i\omega_{ext}t) + \text{c.c.}$, where ω_{ext} is the external driving frequency, chosen here as a reference. The oscillator is placed under the influence of a randomly modulated input signal with amplitude $\eta(t)$, which is considered to be a zero-mean Gaussian noise with correlation function to be defined later. Assuming that the driving is strong and fast, one can write $\eta(t) = \varepsilon^{-1}H(T)$, where $\varepsilon \ll 1$ is a smallness parameter, and $T = \varepsilon^{-1}t$ is the fast time scale. Under this assumption, one can postulate an asymptotic solution to the Stuart-Landau equation as $A(t) = A_0(t, T) + \varepsilon A_1(t, T) + \mathcal{O}(\varepsilon^2)$. Substituting this expansion into Eq. (1), writing $\eta(t)$ in terms of $H(t)$, and comparing the different orders in ε one finds, at $\mathcal{O}(\varepsilon^{-1})$, that A_0 obeys the evolution equation $\partial_T A_0(t, T) = H(T)$, while at $\mathcal{O}(\varepsilon^0)$ one obtains the evolution equation for A_1:

$$\partial_T A_1(t, T) = (1 + i\nu)A_0 - (1 + i\beta)|A_0|^2 A_0 - \partial_t A_0. \quad (2)$$

One can formally integrate the equation obeyed by A_0 to obtain $A_0(t, T) = G(T) + a(t)$, where $G(T) = \int H(t)dt$, and $a(t)$ is a function to be determined that does not depend on the fast time scale T. Taking this expression into account, and examining Eq. (2), one finds that in order for A_1 to be bounded, the following solvability condition must hold:

$$\partial_t a(t) = [(1 - 2\gamma') + i(\nu - 2\beta\gamma')]a - (1 + i\beta)\gamma a^* - (1 + i\beta)|a|^2 a. \quad (3)$$

To obtain this equation one must assume that $\lim_{T\to\infty}[\frac{1}{T} \times \int_0^T G(t)dt] = \lim_{T\to\infty}[\frac{1}{T}\int_0^T |G(t)|^2 G(t)dt] = 0$, and define $\gamma = \lim_{T\to\infty}[\frac{1}{T}\int_0^T G(t)^2 dt]$ and $\gamma' = \lim_{T\to\infty}[\frac{1}{T} \times \int_0^T |G(t)|^2 dt]$. In what follows we assume $G(t)$ real, so that $\gamma = \gamma'$. Equation (3) is identical to the parametrically driven complex Ginzburg-Landau equation that describes the behavior of deterministically rocked systems [8,10]; in the case of harmonic modulation of the driving, for instance [i.e., $\eta(t) = F\cos(\omega t)$, where ω is the rocking frequency], the oscillator becomes locked to one of two stable phases, symmetrical with respect to the driving phase (i.e., rocking arises), provided $\gamma_{min} < \gamma < \gamma_{max}$, with $\gamma = (1/2)(F/\omega)^2$. For $\beta = 0$, which we assume in what follows without loss of generality, the boundaries are given by $\gamma_{min} \equiv |\nu|$ and $\gamma_{max} \equiv (2 + \sqrt{1 - 3\nu^2})/3$. The boundary $\gamma = \gamma_{min}$ corresponds to a saddle-node bifurcation, whereas the one at $\gamma = \gamma_{max}$ is determined by a supercritical pitchfork bifurcation.

The previous conclusion does not require the driving to be periodic. Rocking occurs even if the driving is stochastic (which is the case we are interested in here), provided γ exists and is bounded by the values given above. We now address the issue of how to generate a noise that obeys these conditions.

Numerical simulations.—The values of γ and γ' diverge if $\eta(t)$ is chosen to be white, since $G(T)$ is then a Wiener process, whose square grows linearly with T. This problem would not arise, for instance, if $G(T)$ itself is a white noise, or a correlated noise with small correlation time. We can thus construct the noise $\eta(t)$ in discrete time by generating a Gaussian white noise with zero mean and intensity unity (which in fact corresponds to a colored noise with a small correlation time equal to the finite time step Δt), and numerically differentiating this noise. This results in a noise whose power spectrum increases as $S(\omega) \sim \omega^2$ [violet noise, see Fig. 1(d)], for which γ is well defined and is approximately equal to F^2, where F is the intensity of the original noise $\eta(t)$.

We simulated the evolution of $x(t) = A\exp(i\omega_{ext}t) + \text{c.c.}$, with $A(t)$ obeying the Stuart-Landau Eq. (1) for the noise described above, for which $\gamma = F^2$. The dynamics of the system in phase space is shown in the upper panels of Fig. 1 for two cases: inside and outside the rocking range limited by γ_{min} and γ_{max}. Figure 1(a) clearly reveals the existence of rocking for a value of the noise intensity F inside the rocking range: the phase of the oscillator is locked at a value $\pi/2$ with respect to that of the driving (used here as a reference). Outside of the rocking range, on the other hand, the oscillator is free running [Fig. 1(b)].

Figure 1(c) depicts the phase diagram of the system in the plane formed by the noise intensity F and the detuning ν. The numerically obtained rocking range (bounded by symbols in the figure) agrees reasonably well with that predicted theoretically above, namely $\gamma_{min} < \gamma < \gamma_{max}$ with $\gamma = F^2$. Below the rocking region, random modulation of the driving is not intense enough to produce rock-

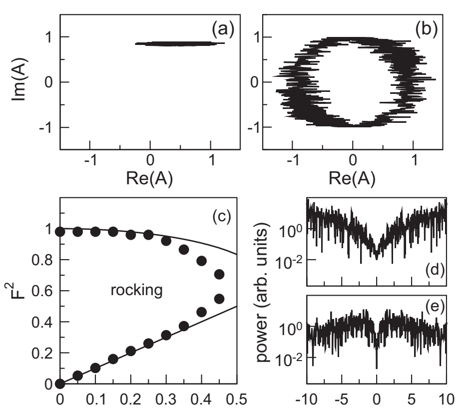

FIG. 1. Theoretical and numerical analysis of noise-induced rocking. The upper plots show phase space trajectories of $x(t)$ for values of the noise intensity F inside (a) and outside (b) of the rocking range. Panel (c) shows the phase diagram of the system. Symbols correspond to numerical simulations and lines to the theoretical rocking boundaries limited by γ_{min} and γ_{max}. The power spectrum of the driving signal is shown in the unfiltered case (d) and after the application of a double Lorentzian filter with cutoff frequency $\Omega = 10$ (e). Parameters are $F^2 = 0.2$, with $\nu = 0.1$ for the rocking case (a) and $\nu = 0.3$ for the free-running case (b). The simulations shown in (a)–(c) correspond to the unfiltered driving of plot (d).

ing; above the upper region, on the other hand, the system operates in an adiabatic regime, following closely the modulation.

We next checked the robustness of the rocking phenomenon with respect to the statistics of the driving signal. To that end, we truncated the driving signal at high frequencies by using a Lorentzian transform function $L(\omega) = \Omega^2/(\Omega^2 + \omega^2)$, with frequency cutoff Ω. If applied once to our violet noise, the resulting power spectrum is $S(\omega) = \Omega^2\omega^2/(\Omega^2 + \omega^2)$; i.e., the filtered noise has the same asymptotic behavior at low frequencies, $S(\omega) \sim \omega^2$, and it saturates at high frequencies, $S(\omega) \sim \omega^0$. If applied twice to the noise, it results in a power spectrum $S(\omega) = \Omega^4\omega^2/(\Omega^2 + \omega^2)^2$; i.e., the filtered noise again has the same asymptotics at low frequencies, but it decays at high frequencies, $S(\omega) \sim \omega^{-2}$. The latter case corresponds more closely with the experiment [compare Fig. 1(e) with the inset of Fig. 4]. In both cases, the rocking was observed in a particular region, depending on the cutoff frequency Ω.

Experimental demonstration.—In order to verify experimentally that noise can induce rocking, and thus bistable phase locking, we use a nonlinear Chua circuit operating in a periodic regime. This circuit is made of standard electric components plus a nonlinear resistor built from operational amplifiers, which leads to rich dynamical behavior including bistability, excitability, and chaos. The circuit and its internal parameters are given in Ref. [10] except for $L = 18$ mH, $R_{\mathrm{exc}} = 1.954$ kΩ, and $R_{\mathrm{coup}} = 21.8$ kΩ (see Fig. 1 of Ref. [10] for details). Under these conditions the circuit has a (natural) frequency $f_0 = 2827.1$ Hz. The external driving is generated by multiplying a pure sinusoidal voltage signal (with frequency f_{ext}) by a Gaussian white noise. The compound signal is then filtered with a bandpass fourth-order Butterworth filter. Next, we introduce the external signal into the Chua circuit through a voltage follower and a coupling resistor (see Ref. [10] for details). The spectrum of the external signal, shown in the inset of Fig. 4 below, exhibits a dip centered at the fundamental frequency ω_{ext} of the external signal.

According to the theoretical analysis presented above, rocking should arise for intermediate noise levels. An example of the phenomenon is shown in Fig. 2. In panels (a),(d) we show the periodic component of the external perturbation shifted by $\pi/2$, which can be used as a reference to observe the two possible locked states. Panels (b),(e) show the complete external perturbation, whose spectrum is shown in the inset of Fig. 4 [cf. Fig. 1(e)]. Finally, panels (c),(f) display the output of the system (voltage of one of the capacitors, see Ref. [10]), for two different realizations of the experiment. A comparison with the reference signal shows that the output signal is (nearly) in phase (c) or in antiphase (f) with the reference signal, and the phase relation is constant in time. The noisy driving signals (middle plots), on the other hand, are shifted with respect to the output (and reference), and

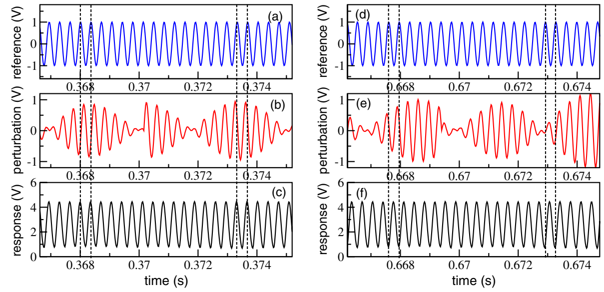

FIG. 2 (color online). Rocking in a nonlinear electronic oscillator. The two columns (a)–(c) and (d)–(f) represent two different realizations of the experiment, leading to two different locking states with respect to a common reference signal. In each column, from top to bottom: periodic component of the external signal (shifted by $\pi/2$), total external signal, and oscillator voltage.

furthermore their phase drifts in time (compare the two pairs of dashed lines). A comparison between plots (c) and (f) of Fig. 2 shows the bistable nature of the rocking phase. To obtain the results of panel (f), we decrease the voltage of the external input until rocking is lost. Then, we increase the input again until we enter once more the rocking region, keeping the reference phase constant. In this case, the output signal is shifted by (nearly) π with respect to the reference signal (which is in turn shifted, we recall, by $\pi/2$ with respect the sinusoidal signal included in the external perturbation).

Figure 3 shows a comparison of the phase in the two situations of plots 2(c) and 2(f). The x axis corresponds to the reference signal and the output is assigned to the y axis. There is a phase shift between the reference signal and the output (already corrected in the figures), introduced by the impedance of the circuit, which can be estimated from the circuit parameters. After correcting the phase shift and normalizing both signals, Fig. 3 shows that the differences of both phases with respect to the reference signal are constant in time and shifted by π with respect to each other.

It is difficult to estimate the region where rocking appears, since it depends on the shape of the spectrum of the

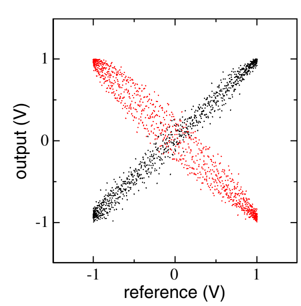

FIG. 3 (color online). Phase relation between the reference signal and the output of the system. Red (or gray) and black lines correspond to the two possible phase relations of Figs. 2(c) and 2(f), respectively.

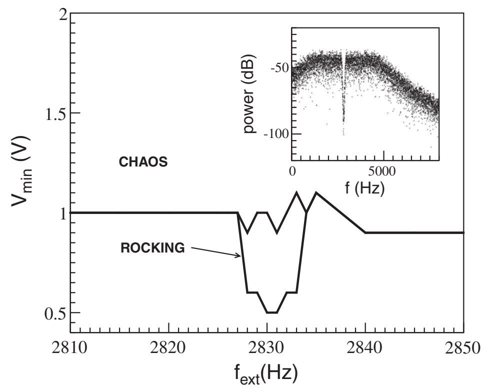

FIG. 4. Rocking region as a function of the fundamental frequency of the external perturbation f_{ext}. V_{min} is the minimum amplitude of the external signal that leads to rocking. The inset shows the power spectrum of the input signal.

input signal and also on its fundamental frequency. Figure 4 shows the range of driving amplitudes where rocking exists, when the fundamental frequency of the driving signal is modified, keeping the shape of the spectrum constant. To obtain the results of Fig. 4, we filter the input signal with a bandpass filter of cutoff frequencies $f_{low} = 100$ Hz and $f_{high} = 2000$ Hz, leading to the spectrum shown in the inset. By modifying the fundamental frequency of the external signal we shift the position of the minimum voltage that guarantees rocking, keeping the shape of the spectrum envelope constant. This leads to the phase diagram of Fig. 4, where a rocking region exists for a certain range of input frequencies and amplitudes. The rocking region begins at the natural frequency of the circuit, $f_0 = 2827.1$ Hz, and extends to higher frequencies. Furthermore, we can see that rocking is only obtained for moderate amplitudes of the external perturbation: when the amplitude becomes large enough (i.e., in the case of strong noise), the oscillator is taken to a region of chaotic behavior. We note also that the phenomenon is robust against phase slips when the circuit operates away from the rocking boundaries.

We also analyzed how the rocking phenomenon changes as a function of the shape of the spectrum, keeping the fundamental frequency f_{ext} of the driving signal fixed. To that end we modified the high cutoff frequency f_{high} of the bandpass filter. Figure 5 shows the rocking region for different spectrum shapes. We can see that rocking is lost when the spectrum is wide enough [Fig. 5(e)], although before reaching that limit, the phenomenon holds for a wide frequency region.

Discussion.—Our results show that bistable phase locking can be induced by noise. The effect works not only with a purely real-valued modulation, as shown here, but also with a "slightly" complex-valued (but phase-squeezed) signal whose amplitude-noise component is much larger than the phase-noise one (a strong enough phase noise destroys the rocking effect). Given the ubiquity of driven nonlinear oscillators in nature, we expect the results reported here to be widely applicable to situations in which the driving signal is randomly modulated.

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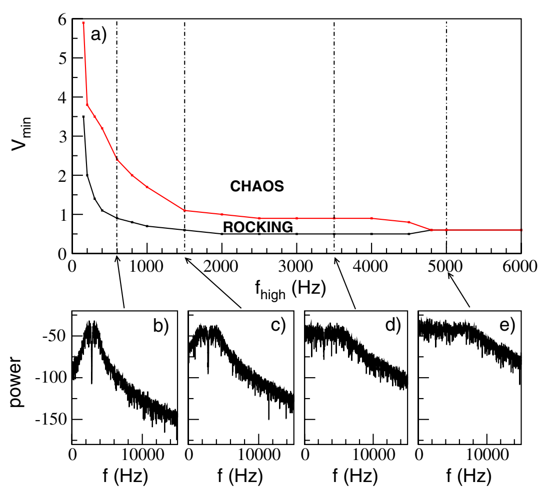

FIG. 5 (color online). Rocking region as a function of the shape of the spectrum. In (a), f_{high} refers to the higher frequency cutoff of the bandpass filter. In the (b)–(e) plots f_{high} is 600 Hz (b), 1500 Hz (c), 3500 Hz (d), and 5000 Hz (e). In (a), the y axis refers to the amplitude (V_{min}) of the external perturbation.

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