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NOISE-ENHANCED WAVE TRAIN PROPAGATION IN UNEXCITABLE MEDIA

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The influence of spatiotemporal colored noise on wave train propagation in nonexcitable media is investigated. This study has been performed within the framework of the Oregonator model in terms of the characteristic noise parameters. Some features seen in single front propagation, like noise induced propagation facilitation for an optimal level of the noise intensity, are also found for periodic wave trains. The main new effect is, however, an enhancement of propagation for correlation times of the noise of the order of the period of the wave train.

1. Introduction

Recently, several publications have come up regarding the effects that noise induces on spatial structures in dynamical systems [Jung *et al.*, 1998; Pérez-Muñuzuri *et al.*, 2000; Sendiña-Nadal *et al.*, 1998, 1999, 2000; Kádár *et al.*, 1998; Wang *et al.*, 1999]. Most of them have dealt with *single front* propagation in *subcritical* conditions, which is enhanced or supported when an optimal *level* of noise is added to the system. This phenomenon raises interest from both the biological and physical parts, since it implies that noisy backgrounds would favor weak signal transmission through neural fibers [Wiesenfeld & Moss, 1995; Moss *et al.*, 1994] or arrays of electronic circuits [Löcher *et al.*, 1998; Lindner *et al.*, 1998]. The phenomenon lying behind these examples is what has been called "stochastic resonance" [Gammaitoni *et al.*, 1998]. Nowadays, this concept is much wider than initially and not all the usual ingredients (a bistable system periodically forced in the presence of noise) are essential.

In 1998, Kádár *et al.* described noise-supported traveling waves in two-dimensional *subexcitable* media. Spatiotemporal noise was applied to a photosensitive chemical medium for different pixel sizes and updated using a Gaussian noise at regular time intervals. An optimal noise level was found at which the relative signal strength became maximal. They also suggested the existence of an optimal noise *timescale* supporting propagation. In subcritical conditions of excitability, there are two different modes of propagation: *subexcitable* [Kádár *et al.*, 1998; Wang *et al.*, 1999; Sendiña-Nadal *et al.*, 2000] and *unexcitable*. In the *sub*excitable regime, wave segments with free ends contract tangentially and may eventually disappear, depending on their size and shape. In the *un*excitable one, any initial perturbation decays in amplitude until it eventually disappears. This last scenario is more dramatic since even unbounded waves disappear.

We study the propagation of a one-dimensional *train* of wave fronts in the unexcitable regime under a spatiotemporal noise forcing. Because of the one-dimensional situation we are considering, only transitions from excitable to unexcitable regimes are possible. This is shown in Fig. 1 where

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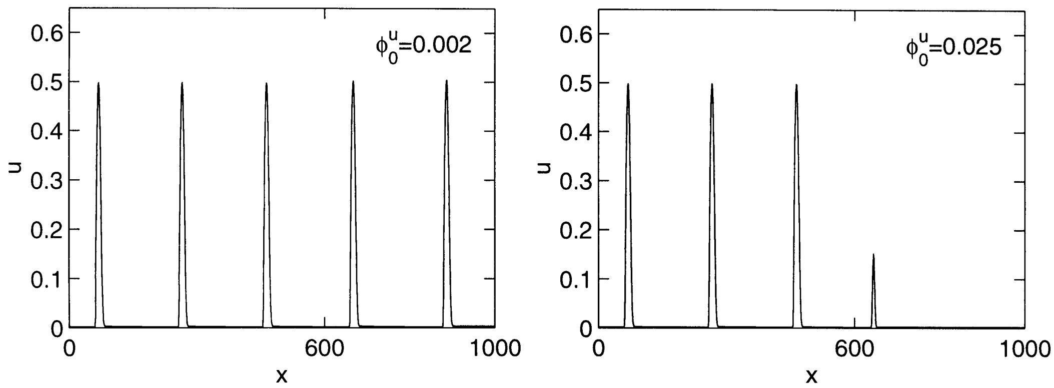

Fig. 1. (Left) Excitable versus (right) unexcitable behavior. In the rightmost figure a wave transmission failure is shown where the traveling pulse decreases in amplitude to the point of extinction.

propagation failure occurs when the control parameter is below some excitability threshold.

The consideration of a time periodic structure introduces a new feature: the possibility of interaction between consecutive fronts. We have found different rates of supported transmission depending on the period of the wave train and on the correlation time of the noise. In this paper, we study a new phenomenon related to stochastic resonance consisting in the enhancement of the propagation of a *periodic* wave train through an unexcitable channel, as a consequence of the time-correlation of spatiotemporal noise. An overall enhancement of wave propagation in these conditions is achieved for an optimal combination of noise *intensity*, correlation *time* and correlation *length*.

2. Numerical Model

Numerical studies were performed using a two-variable Oregonator model in one dimension

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{\varepsilon}\left\{u - u^2 - [fv + \phi(x,\,t)]\,\frac{u-q}{u+q}\right\} \\ &\quad + D_u\,\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= u - v\end{aligned} \tag{1}$$

where u is the activator and v the inhibitor. D_u is a diffusion coefficient, and f, q and ε are parameters related to the Belousov–Zhabotinsky (BZ) kinetics. Equations (1) were numerically integrated using an Euler method [Sancho *et al.*, 1982] with a time step of 10^{-3} t.u. and a grid size of 0.15 s.u. in an array of $N = 1000$ points. Zero flux boundary conditions were considered at the end of the system $x = N$. The spatiotemporal fluctuations were introduced through the light intensity parameter ϕ, that accounts for the photosensitivity of the BZ reaction catalyzed by ruthenium [Krug *et al.*, 1990]. Under homogeneous illumination, the system becomes unexcitable for $\phi > 0.02$. Pulses of constant amplitude $A = 0.2$ and width $\delta t = 0.1$ t.u. (equal to 100 time steps) were periodically delivered at $x = 0$, in order to obtain a wave train with constant period T. Waves develop and travel through an excitable medium (its properties being determined by the value of $\phi_0^e = 0.002$) of size $N_u = 600$ points before entering the unexcitable region, consisting of an array of independent fluctuating cells of size ℓ lattice units, with average light intensity $\phi_0^u = 0.025$ (see Fig. 2). The expression for the fluctuating field as a function of the position is thus

$$\phi(x,\,t) = \begin{cases} \phi_0^e & 0 < x < N_u \\ \phi_0^u + \displaystyle\sum_{i=1}^{n_l} \xi_i(t)\Theta_i(x) & N_u \le x \le N \end{cases} \tag{2}$$

where i is the discrete coordinate of a noise cell ranging from 1 to $n_l = (N - N_u)/\ell$ and $\Theta_i(x)$ is a shorthand for $\Theta_i(x) = \Theta(x - (i-1)\ell)\,\Theta(i\ell - x)$, Θ being the Heaviside function. $\xi_i(t)$ stands for an Ornstein–Uhlenbeck process at cell i, namely a Gaussian process with zero mean and a correlation function $\langle \xi_i(t)\xi_l(t')\rangle = \sigma^2 \exp(-|t-t'|/\tau)\delta_{il}$ [García-Ojalvo & Sancho, 1999]. τ denotes its correlation time and σ^2 is the noise variance. The parameter ℓ fixes the characteristic length of the inhomogeneous fluctuating excitability. In the limit $\tau \to 0$ the white-noise limit $\xi_w(t)$ is recovered if

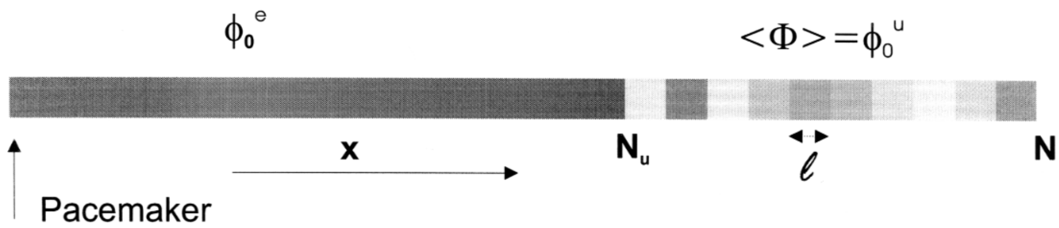

Fig. 2. Simulated medium with the excitability represented in a eight-bit gray scale. N_u denotes the boundary between the excitable region on the left, where the wave train is created and propagates, and the spatial distribution of cells of size ℓ that fluctuate independently of one another around a mean value ϕ_0^u in the inhibition region.

$\sigma^2\tau$ is finite. For $\tau \to \infty$ the *frozen* or static Gaussian distributed noise is obtained. Numerical simulations have been carried out by varying the noise dispersion σ, the correlation time τ and the noise pixel size ℓ. Noise was solved using an integral algorithm described in [Fox *et al.*, 1988] and the Gaussian random number generator proposed by [Toral & Chakrabarti, 1993] was used in the simulations. Different control points were equally distributed (each 25 lattice units) along the unexcitable channel, and another one was placed in the excitable zone ($x = 500$). Measurements of the ratio between the number of wave fronts reaching each control point and that of those exiting the excitable region were performed and averaged over five realizations.

3. Results

Without noise ($\sigma = 0$) waves entering the subexcitable region immediately die out, while depending on the noise parameters waves can travel longer distances. This distance depends as well on the wave train period T. Figure 3 shows the percentage of wave fronts reaching each control point beyond the border between the excitable and the unexcitable regions with and without noise. Note the exponential decay with x ($x > N_u$) when noise is present, compared to the abrupt (linear) fall when it is absent, indicating that a few fronts are able to survive and do indeed reach very far. On the other hand, it seems that more spaced fronts ($T > 5$) are more vulnerable to noise, since the distance to which all of them arrive is drastically reduced.

The main results of the numerical simulations performed varying the different noise parameters are summarized in Fig. 4 for a constant value of $T = 5$ t.u. A colormap plot represents the percentage of wave fronts reaching the control point situated at 175 points beyond the border ($N_u = 600$) as a function of the noise dispersion σ and the correlation time τ, for three different values of the noise pixel size ℓ. From the figures it is evident that *wave propagation depends on the spatial correlation of the noise.* As ℓ is increased, the percentage of wave fronts reaching the control position increases as well (note the different scales on the colorbars) and they reach a maximum for higher values of the correlation time.

- For $\ell = 10$, the noise pixel size is slightly smaller than the front width (≈ 20 lattice spacings) and there is no noise level or time scale for the noise that appears to be optimal. Instead, up to a critical value of the noise intensity (below which the transmission rate is zero) there is a sharp increase that saturates for high values of σ. No clear dependence on the correlation time is observed.
- For $\ell = 100$ there are large highly correlated regions, in such a way that each front propagates under almost pure temporal noise. So for low correlation times, temporal fluctuations occur so frequently that they are averaged out by the medium. On the other hand, for $\tau > T$ the noise varies slowly and there are fluctuations that allow a more lasting propagation.
- Only for $\ell = 40$ there is a compromise between noise intensity and noise correlation time, giving rise to a tiny improvement on wave propagation for intermediate values of σ and for correlation times of the order of the period of the wave train $T = 5$.

The global effect of increasing the spatial correlation is thus a shift to the right of the maximum for the transmission rate, here shown as a red area in Fig. 4.

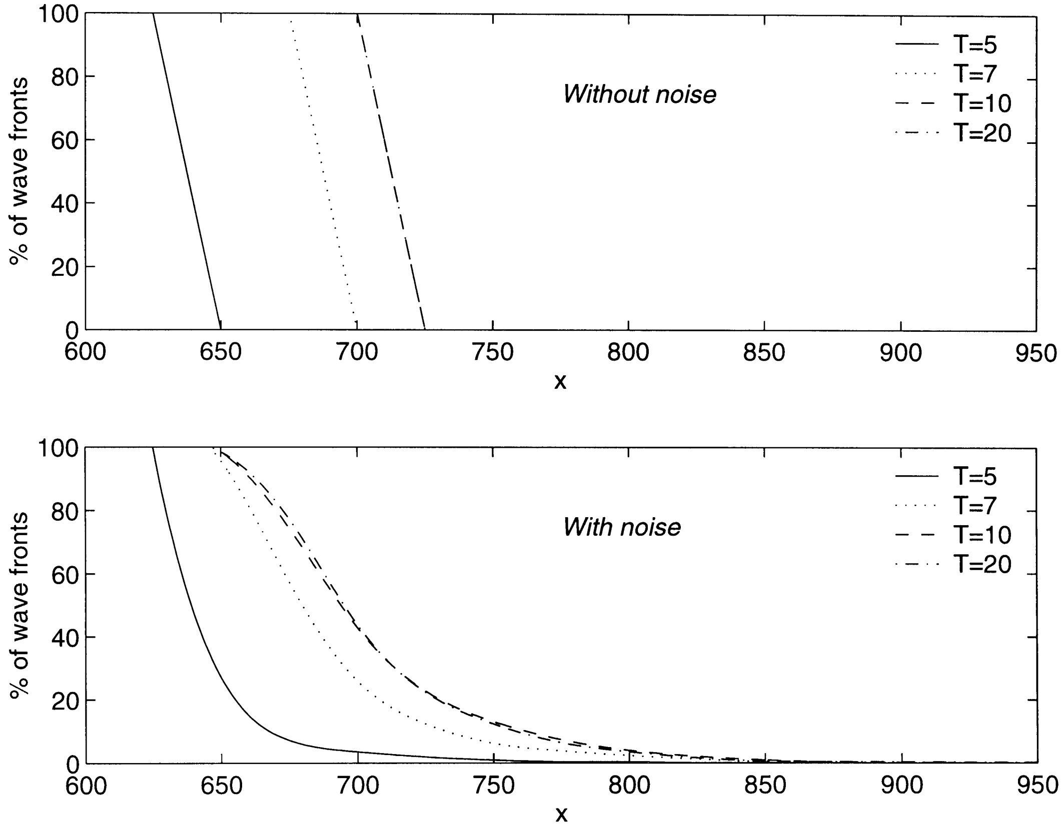

Fig. 3. Percentage of wave fronts reaching the different control points placed along the unexcitable region ($x > N_u$), (bottom) with and (top) without noise, for several periods of the wave train. Numerical parameters: $\varepsilon = 0.05$, $f = 3$, $q = 0.002$, $D_u = 1$. Noise parameters: $\sigma = 0.001$, $\log_{10} \tau = -0.75$, $\ell = 40$ points.

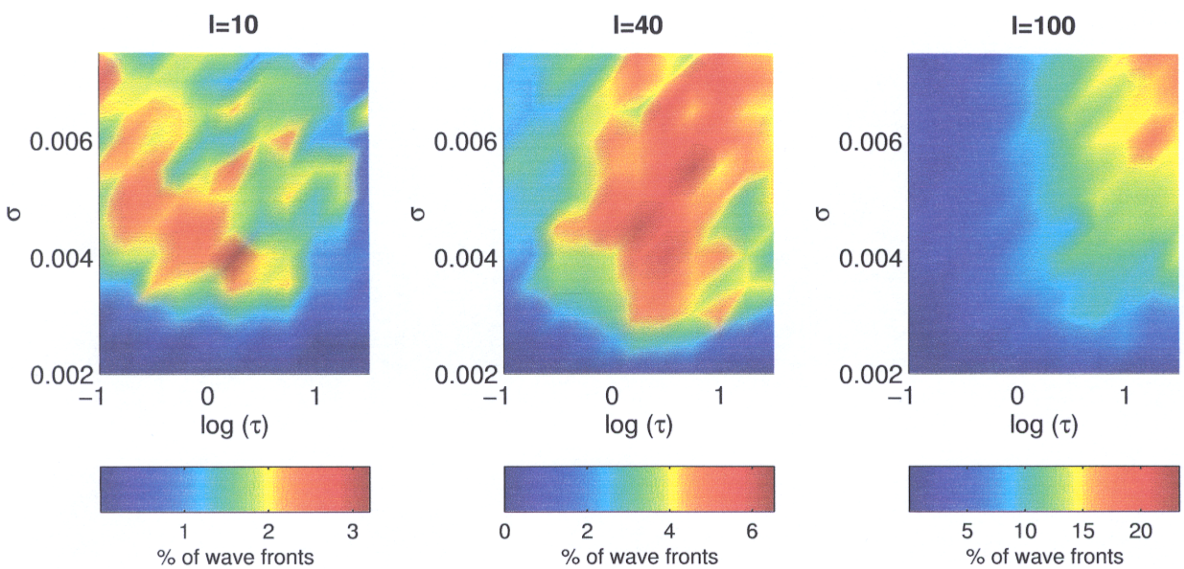

Fig. 4. Colormap plots of the percentage of wave fronts reaching the control position at $x = N_u + 175$, as a function of the noise dispersion σ and correlation time τ for three different noise pixel sizes ℓ. Note the different scales on the percentage of wave fronts of the colorbars. Numerical parameters: $\varepsilon = 0.05$, $f = 3$, $q = 0.002$, $D_u = 1$ and $T = 5$.

For $\ell = 40$ and a constant value of τ (≈ 1) there occurs stochastic resonance as a function of the noise amplitude. A better perspective of the phenomenon is given in Fig. 5, for $T = 20$ t.u. and at different control points. Close to the boundary N_u the transmission rate decreases as σ increases, see Fig. 5(a). But far away from the extinction point for the deterministic unexcitable system ($x > 125$) the rate of wave fronts that reach distant positions exhibits a peak as a function of the noise variance σ^2 (the signature of a stochastic resonance) Fig. 5(b).

A resonance-like behavior is also observed with respect to the correlation time. Figure 6 shows the percentage of wave fronts versus τ for different

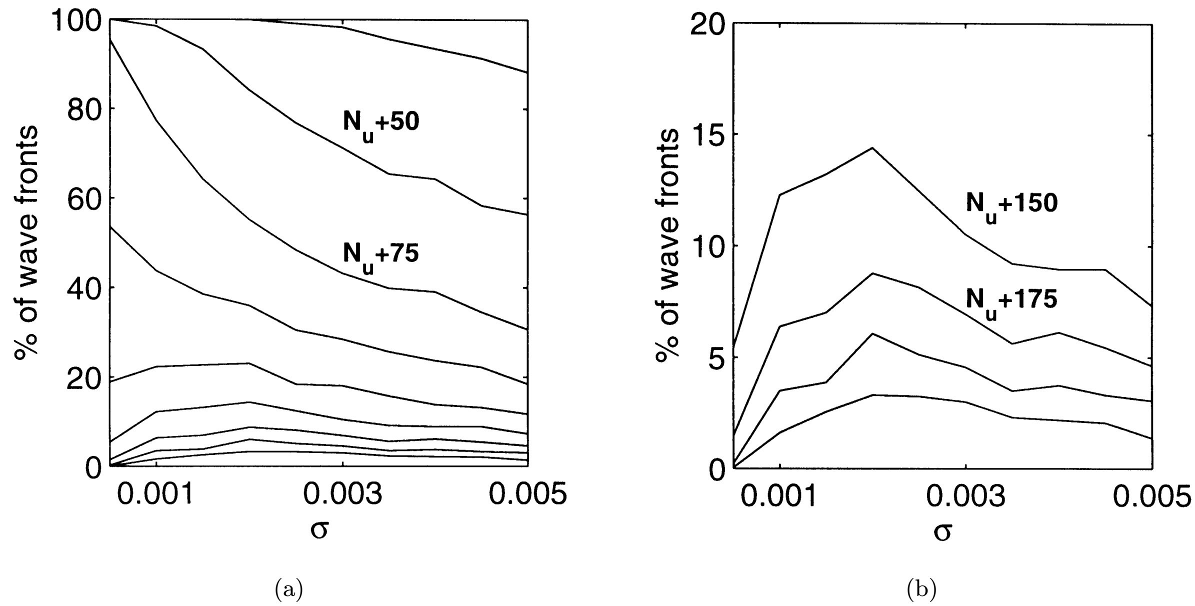

Fig. 5. Percentage of wave fronts reaching the positions indicated as labels superimposed to the graphs, as a function of the noise dispersion σ, for $T = 20$ t.u., $\ell = 40$ and $\log(\tau) = -0.75$. (b) Corresponds to a magnification of (a).

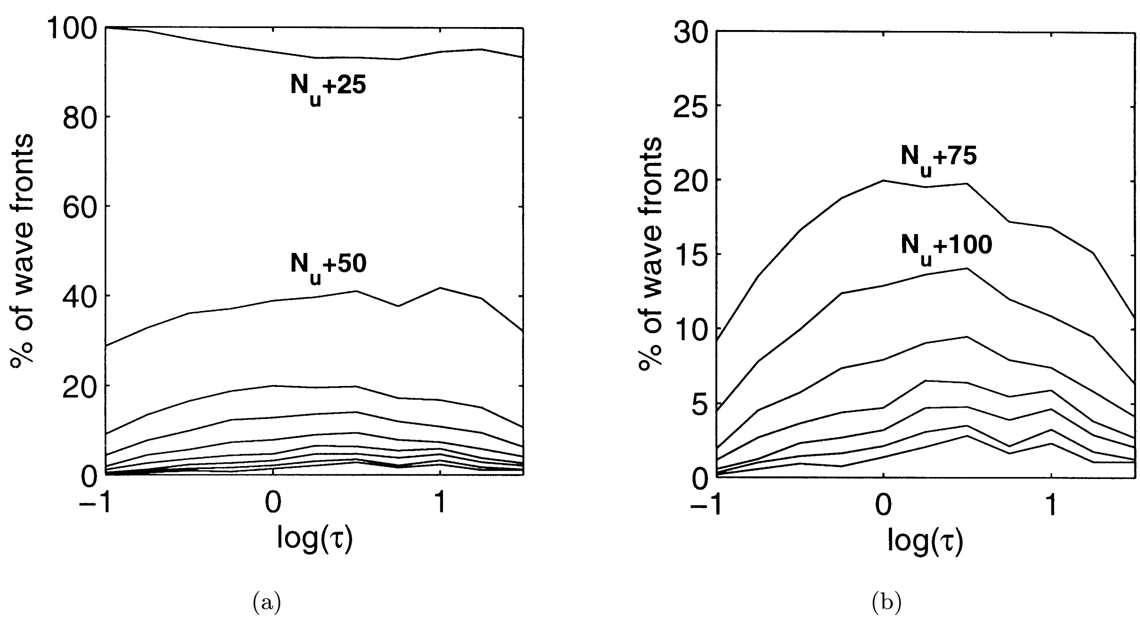

Fig. 6. Percentage of wave fronts reaching the positions indicated as labels superimposed to the graphs, as a function of the correlation time τ, for $T = 5$ t.u., $\ell = 40$ and $\sigma = 0.001$. (b) Corresponds to a magnification of (a).

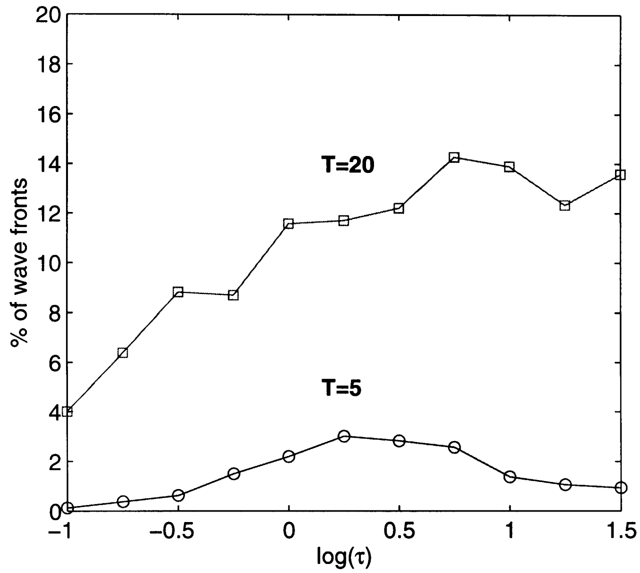

Fig. 7. Behavior of the rate of collected wave fronts reaching position $x = N_u + 175$ as a function of the correlation time τ, for two different periods of the wave train T. Noise parameters: $\sigma = 0.001$ and $\ell = 40$.

control positions at constant noise intensity σ. As in the previous case, a local maximum develops as τ increases [Fig. 6(b)], while for $\tau \to 0$ the medium averages fluctuations and for $\tau \to \infty$ the wave train has to overcome a static structured noise.

In order to gain an insight into the meaning of this last behavior, we have represented the percentage of wave fronts reaching a fixed position ($x = N_u + 175$) for two different delivering frequencies of the pulses, as a function of τ. The result is plot in Fig. 7. It can be observed that the maximum rate occurs at different values of τ and moreover that this value is of the order of the period of the wave train.

Summing up, we have investigated the effect of spatiotemporal fluctuations on a wave train propagating in an unexcitable regime and shown that stochastic resonance occurs not only for an optimal intensity of the noise but also for a correlation time that matches the characteristic time of the periodic structure. Globally, the introduction of noise extends the propagation length. This is in agreement with other studies in noisy overdamped bistable oscillators [Lindner *et al.*, 1998; Perazzo *et al.*, 2000; García-Ojalvo *et al.*, 2000] where the interplay among noise, nonlinearity and forcing gives rise to an enhancement of propagation.

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