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INTERACTING OSCILLATORS IN COMPLEX NETWORKS: SYNCHRONIZATION AND THE EMERGENCE OF SCALE-FREE TOPOLOGIES

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In natural systems, many processes can be represented as the result of the interaction of self-sustained oscillators on top of complex topological wirings of connections. We review some of the main results on the setting of collective (synchronized) behaviors in globally and locally identical coupled oscillators, and then discuss in more detail the main formalism that gives the necessary condition for the stability of a synchronous motion. Finally, we also briefly describe a case of a growing network of nonidentical oscillators, where the growth process is entirely guided by dynamical rules, and where the final synchronized state is accompanied with the emergence of a specific statistical feature (the scale-free property) in the network's degree distribution.

Keywords: Synchronization; coupled oscillators; complex networks.

1. Introduction: Coupled Oscillators in Lattices and Regular Networks

The recent years have witnessed an increasing interest from the scientific community toward the study of the dynamics in complex networks, i.e. ensembles of interacting dynamical elements whose connectivity structure is irregular, complex and possibly evolving in time [Albert & Barabási, 2002; Newman, 2003; Boccaletti *et al.*, 2006].

A massive analysis of networks from the real world has allowed to unravel a series of unifying principles and statistical properties common to such connecting topologies. Probably, the most important of them is the scaling of their degree distribution. The degree distribution, $P(k)$, defined as the probability that a node chosen uniformly at random has degree k, has been found to ubiquitously and significantly deviate from the Poisson or Gaussian distributions expected from random graphs and, in many cases, to exhibit a power law (scale-free, SF) tail (i.e. $P(k) \sim k^{-\gamma}$) with an exponent γ taking a value between 2 and 3. Together with this, real world networks are generally characterized by the *small world* property, implying that their average shortest path (the average distance of two nodes of the network along their shortest path) scales at most logaritmically with the system size.

Later, attention has been diverted to understand the intimate relationship between the topological structure displayed by a graph, and the mechanisms leading to the arousal of a collective behavior (as e.g. the synchronization of all the network's nodes into a common dynamical behavior).

Here, we discuss some results about the emergence of such synchronized behavior in complex networks of interacting self-oscillatory systems, and particularly its intimate relationship with the emergence of those specific topological structures that are, indeed, encountered in most of the natural networking systems.

When the study of interacting self-sustained oscillators was first approached, three main coupling configurations were considered, namely *global coupling*, when each unit interacts with all the others; *local coupling*, when elements interact only with their neighbors (defined by a given metric); and *nonlocal or intermediate couplings*.

In an ensemble of globally coupled oscillators, each individual unit is generally considered as influenced by the global dynamics through an interaction of *mean-field* type. In these kinds of configurations, for both limit-cycle and chaotic oscillators with slightly different oscillation modes, a phase transition associated to a collective and coherent behavior was observed as the result of an increase in the coupling strength [Kuramoto, 1984; Kuramoto & Shinomoto, 1985; Pikovsky *et al.*, 1996; Kiss *et al.*, 2002].

However, a global coupling architecture is not always able to properly describe a real situation, especially when the units are embedded in a physical array, or when the space dimension is a relevant feature of the system that one wants to model. For instance, in some situations, such as neural networks or power supply networks, one should consider that long range connections (necessary to the all-to-all interactions) may imply a high cost in terms of energy.

In a series of complimentary approaches, the oscillatory interacting systems were embedded in D-dimensional lattices. Not always though this latter approach allows for its rigorous analysis [Strogatz, 2000; Strogatz & Mirollo, 1988; Matthews *et al.*, 1991], different collective regimes were observed (global or partial synchronization, antiphase synchronization, phase clustering, etc.) in large ensembles of coupled chaotic or periodic elements. In almost all cases, such cooperative dynamics are strongly dependent on the size and dimension of the lattices, as well as on the distribution of the eigenfrequencies of each oscillator [Sakaguchi *et al.*, 1987; Klevecz *et al.*, 1992; Osipov *et al.*, 1997; Zheng *et al.*, 1998; Belykh *et al.*, 2000; Zhang *et al.*, 2001; Liu *et al.*, 2001; Zhou & Kurths, 2002].

Later, a series of nonlocal coupling schemes were proposed to take into account the natural decay of the information content with the distance, especially to model the dependence of the spatial correlation on the range of nonlocal coupling [Kuramoto, 1995; Kuramoto & Nakao, 1996, 1997; Rogister *et al.*, 2004]. In real physical systems, moreover, a certain degree of randomness usually exists not only in the intrinsic properties of each unit, but also in the connections between them, and these *natural* disorders (mismatch of parameters) may strongly affect the collective behaviors in both global or local coupling schemes.

Most studies on coupled oscillators considered N coupled phase oscillators, derived from the Kuramoto model [Kuramoto, 1984]:

$$\dot{\phi}_i = \omega_i - \sigma \sum_{j=1}^{N} \mathcal{C}_{ij} \sin[\phi_j - \phi_i], \quad i = 1, \dots, N, \tag{1}$$

where ϕ_i denotes the phase of the ith oscillator, ω_i its natural frequency, σ is the coupling strength parameter, and $\mathcal{C}_{ij} = 1$ if nodes i,j are connected, and $\mathcal{C}_{ij} = 0$ otherwise (i.e. $\mathcal{C}_{ij}$ are the elements of the so-called adjacency matrix associated to the network of the interacting oscillators).

In general, the order parameter to monitor the appearance of a synchronized motion in Eq. (1) is the so-called r parameter, defined by

$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j(t)} \right| \tag{2}$$

which gives the time dependent value of the ensemble average of all vectors corresponding to the unit circle representation of the oscillators' phases. The parameter, therefore has a value close to 0 whenever the network is phase incoherent, and equal to 1 when it is phase synchronized.

The original Kuramoto model [Kuramoto, 1984] corresponds to the simplest case of globally coupled (full connected graph) with equally weighted oscillators, where $\mathcal{C}_{ij}$=1, $\forall\, i \neq j$. In the original approach, the coupling strength was taken to be $\sigma = \varepsilon/N$ (with $\varepsilon > 0$, i.e. it was a coupling parameter normalized to the system size) in order to warrant the smoothness of the model behavior also in the thermodynamic limit $N \to \infty$ [Kuramoto, 1984; Strogatz, 2000]. It was pointed out that, in this case, the onset of synchronization of the oscillators' phases and frequencies occurs at a specific critical value of the coupling strength.

As for locally coupled oscillators, [Niebur *et al.*, 1991] considered the phase locking process in a lattice array for different topologies: regular nearest neighbors coupling, random-local coupling (where the weights of connections $\mathcal{C}_{ij}$ were randomly drawn from a Gaussian distribution), and long-range sparse connections. The main observation was that, at the same overall coupling strength, long-range interactions produce a faster and more robust synchronization than local coupling topologies.

Among the pioneering studies conducted on the effects of synchronization of oscillators in complex networks, we mention the numerical analysis of the Kuramoto model on top of small-world networks [Watts, 1999], and the study of the analytical conditions for complete synchronization of chaotic systems on different kinds of graphs [Barahona & Pecora, 2002].

These first attempts were aimed to give a suitable representation to a series of situations, for instance in biology, where it is useful to consider the nodes of a given network as oscillatory systems. Examples are ensembles of coupled and pulse-coupled oscillators with and without time delay, widely used because of their relevance to natural systems such as chirping crickets and flashing fireflies, among others [Yeung & Strogatz, 1999; Timme *et al.*, 2002].

Later, several groups turned to investigate the synchronization phenomena of the Kuramoto model in complex wirings. In particular, Moreno *et al.* [Moreno & Pacheco, 2004; Moreno *et al.*, 2004] studied numerically the conditions for the *onset* of synchronization in scale-free networks, and on top of motifs (small subgraphs) that were relevant in different biological and social networks, with the aim of inspecting the critical point associated to the onset of synchronization. The main result of these studies was to show that the onset of synchronization for scale-free networks occurs at a small, though nonzero, value of the coupling strength, and that such a critical point does not depend on the system size N. Several other authors [Lee, 2005; Ichinomiya, 2004] investigated the same problem from a theoretical perspective, as well as with numerical simulations. The same qualitative behavior was reported in [Ichinomiya, 2004], where a mean field theory [Lee, 2005; Ichinomiya, 2004] predicted that the critical point is determined by the all-to-all Kuramoto value, σ_0, rescaled by the ratio between the first two moments of the degree distribution, $\sigma_{mf} = \sigma_0(\langle k\rangle/\langle k^2\rangle)$.

We have organized the paper as follows. In the first part we will discuss some results concerning the stability of the synchronous state in a network of interacting identical oscillators, and, in particular, we will concentrate on the formalism called *the Master Stability Function*, which gives necessary conditions to establish a synchronous motion in an arbitrary network of coupled identical oscillatory systems.

In the second part, instead, we will consider a network of nonidentical phase oscillators to show how the emergence of a synchronous behavior is accompanied with the setting of a specific statistics in the degree distribution, when the network is initially set in a unsynchronized state, and later grown by means of subsequent links (established following a purely dynamical rule) to an external pacemaker that has the role of entraining the pristine nodes' phases.

2. Identical Oscillators in Complex Networks: The Master Stability Function

In this first section, we consider the case of a network of identical oscillators, and we illustrate the so-called Master Stability Function approach. The method, that was originally introduced for arrays of coupled oscillators [Pecora & Carroll, 1998; Fink *et al.*, 2000], has been, indeed, later extended to complex networks with arbitrary topologies [Barahona & Pecora, 2002; Chen *et al.*, 2003; Hu *et al.*, 1998; Zhan *et al.*, 2000; Belykh *et al.*, 2004].

In order to do so, we will consider a generic network of N coupled oscillators. The state of the ith oscillator is represented by a m-dimensional vector field $\mathbf{x}_i \in \mathbb{R}^m$, whose evolution is ruled by an ordinary differential equation $\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i)$, being $\mathbf{F}_i(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}^m$ a function that governs the local (uncoupled) dynamics of the ith oscillator. Then, the resulting equation of motion when the oscillators are coupled reads:

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij}\mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots, N. \tag{3}$$

Here $\mathbf{H}(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}^m$ is a generic vectorial output function giving the signal that is transmitted from an oscillator to the other in the network, σ is the coupling strength, and $\mathcal{L}_{ij} \in \mathbb{R}$ are the elements of a zero row-sum ($\sum_j \mathcal{L}_{ij} = 0,\ \forall i$) $N \times N$ symmetric matrix $\mathcal{L}$ with strictly positive diagonal terms ($\mathcal{L}_{ii} > 0,\ \forall i$), that specifies the strength and topology of the underlying connection wiring.

A rigorous analytic treatment of Eq. (3) requires a series of assumptions. The first is that the network is made of *identical* oscillators, i.e. the evolution function $\mathbf{F}_i$ in Eq. (3) is the same for all network nodes ($\mathbf{F}_i(\mathbf{x}_i) \equiv \mathbf{F}(\mathbf{x}_i),\ \forall i$).

Such an assumption is, indeed, crucial to ensure the existence of an invariant set $\mathbf{x}_s(t)$ in which $\mathbf{x}_i(t) = \mathbf{x}_s(t), \forall i$, representing the complete synchronization manifold $\mathcal{S}$, whose stability will be the object of the subsequent analysis.

The invariance of the synchronization manifold $\mathcal{S}$ (i.e. the fact that it does not depend on σ) is warranted by the zero row-sum condition of the coupling matrix $\mathcal{L}$ as well as by the identity of the coupling function $\mathbf{H}(\mathbf{x})$ for all network's oscillators. These two properties, indeed, cause the coupling term to vanish exactly on $\mathcal{S}$ for all values of the coupling strength, and therefore stability of the synchronous state reduces to take care of the system's dynamical properties along all directions in phase space that are *transverse* to this manifold.

For the sake of clarity, let us start by focusing the attention to the case of symmetric (thus diagonalizable) matrices $\mathcal{L}$.

In this case, the set λ_i of real eigenvalues associated to the orthonormal eigenvectors $\mathbf{v}_i$ that diagonalize the coupling matrix, verifies

$$\mathcal{L}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

and

$$\mathbf{v}_j^T \cdot \mathbf{v}_i = \delta_{ij}.$$

Furthermore, due to the zero-row sum condition, one has that:

(i) the spectrum is entirely semi-positive, i.e. $\lambda_i \geq 0,\ \forall i$, thus they can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$;

(ii) $\lambda_1 = 0$ with associated eigenvector

$$\mathbf{v}_1 = \frac{\pm 1}{\sqrt{N}}\{1, 1, \ldots, 1\}^T$$

that entirely defines the synchronization manifold $\mathcal{S}$, and

(iii) all the other eigenvalues λ_i ($i = 2, \ldots, N$) have associated eigenvectors $\mathbf{v}_i$ spanning all the other directions of the $m \times N$-dimensional phase-space transverse to $\mathcal{S}$.

The necessary condition for the stability of the synchronization manifold is that the set of $(N-1)*m$ Lyapunov exponents that corresponds to phase space directions transverse to the m-dimensional hyperplane

$$\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N = \mathbf{x}_s$$

be all negative values.

Let, therefore

$$\delta\mathbf{x}_i(t) = \mathbf{x}_i(t) - \mathbf{x}_s(t) = (\delta x_{i,1}(t), \ldots, \delta x_{i,m}(t))$$

be the deviation of the ith vector state from the synchronization manifold, and consider the $m \times N$ column vectors

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)^T$$

and

$$\delta\mathbf{X} = (\delta\mathbf{x}_1, \ldots, \delta\mathbf{x}_N)^T.$$

Then one has

$$\delta\dot{\mathbf{X}} = [\mathbb{I}_N \otimes \mathrm{J}\mathbf{F}(\mathbf{x}_s) - \sigma\mathcal{L} \otimes \mathrm{J}\mathbf{H}(\mathbf{x}_s)]\delta\mathbf{X}, \quad (4)$$

where $\otimes$ stands for the direct product between matrices, J denotes the Jacobian operator, and $\mathbb{I}_N$ is the identity matrix.

One notices that the arbitrary state $\delta\mathbf{X}$ can be written as

$$\delta\mathbf{X} = \sum_{i=1}^{N} \mathbf{v}_i \otimes \zeta_i(t)$$

with $\zeta_i(t) \in \mathbb{R}^m = (\zeta_{1,i}, \ldots, \zeta_{m,i})$. Then, by applying $\mathbf{v}_j^T$ to the left side of each term in Eq. (4), one finally obtains a set of N variational equations for the vectors $\zeta_i(t)$ that read

$$\frac{d\zeta_j}{dt} = \mathbf{K}_j\zeta_j,\ j = 1, \ldots, N \quad (5)$$

being

$$\mathbf{K}_j = \mathrm{J}\mathbf{F}(\mathbf{x}_s) - \sigma\lambda_j \mathrm{J}\mathbf{H}(\mathbf{x}_s)$$

specific evolution kernels.

Each equation in (5) yields a set of m conditional Lyapunov exponents. We already discussed that the eigenvalue $\lambda_1 = 0$ has a corresponding eigenmode that lies entirely within the synchronization manifold. Therefore, the corresponding m conditional Lyapunov exponents will be those of the single uncoupled system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ and no conditions on them have to be imposed.

Replacing $\sigma\lambda_i$ by ν in Eq. (5), one can write a parametric variational m-dimensional equation:

$$\dot{\zeta} = \mathbf{K}_\nu \zeta = [\mathbf{J}\mathbf{F}(\mathbf{x}_s) - \nu \mathbf{J}\mathbf{H}(\mathbf{x}_s)]\,\zeta, \qquad (6)$$

from which one can extract all m conditional Lyapunov exponents at each value of the parameter ν.

The parametrical behavior of the largest of such exponents, $\Lambda(\nu)$, is called *Master Stability Function*.

For $\nu > 0$, one has to distinguish between three possible behaviors of $\Lambda(\nu)$ that define three possible classes for the choice of a local function $\mathbf{F}(\mathbf{x})$ and a coupling function $\mathbf{H}(\mathbf{x})$:

(I) $\Lambda(\nu)$ is a monotonically increasing function,
(II) $\Lambda(\nu)$ is a monotonically decreasing function that intercepts the abscissa at some $\nu_c \geq 0$, and
(III) $\Lambda(\nu)$ is a V-shaped function admitting negative values in some range $0 \leq \nu_1 < \nu_2$.

Let us now discuss the relevant properties in these three cases.

Case I corresponds to a choice of $\mathbf{F}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ such that one never achieves synchronization in a network. Indeed, for all possible values of σ and for all possible eigenvalues' distributions (these last ones reflect possible topological arrangements of the wiring connection of the network), the product $\sigma\lambda_i$ always leads to a positive maximum Lyapunov exponent, and therefore the synchronization manifold $\mathcal{S}$ is always transversally unstable.

The very opposite situation arises for functions $\mathbf{F}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ giving Master Stability curves of class II. In these cases, indeed, the network admits synchronization always for a large enough coupling strength, regardless of the topology of the coupling configuration: given any eigenvalue distribution it is indeed sufficient to select $\sigma > \nu_c/\lambda_2$ (where ν_c is the intersection point of the Master Stability Function with the ν axis) to warrant that all transverse directions to $\mathcal{S}$ have associated negative Lyapunov exponents.

In class II, therefore, the effect of the connection topology is to rescale (by means of the second smallest eigenvalue λ_2 of the coupling matrix $\mathcal{L}$) the threshold for the appearance of a synchronous state.

A nontrivial and interesting situation is class III. Here, $\Lambda(\nu)$ is negative in a finite parameter interval (ν_1, ν_2) (with $\nu_1 = 0$ when $\mathbf{F}(\mathbf{x})$ supports a periodic motion). The stability condition is then met for some σ when $\lambda_N/\lambda_2 < \nu_2/\nu_1$.

Systems belonging to class III Master Stability Function are therefore extremely sensitive to specific topological arrangements. It makes sense then to introduce the *network capability* to give rise to a synchronized dynamics as the ratio λ_N/λ_2 between the largest and the second smallest eigenvalues in the spectrum of the coupling matrix. The more packed the eigenvalues of $\mathcal{L}$ are, the higher is the chance of having all Lyapunov exponents into the stability range for some σ [Barahona & Pecora, 2002].

The situation is a bit more complicated when the coupling matrix $\mathcal{L}$ is asymmetric (but still diagonalizable). In this case, the spectrum of $\mathcal{L}$ is contained in the complex plane ($\lambda_1 = 0$; $\lambda_l = \lambda_l^r + i\lambda_l^i$, $l = 2, \ldots, N$), and one has to study the parametric equation (6) for complex values of the parameter $\nu = \nu^r + i\nu^i$.

An ordering of $\mathcal{L}$'s eigenvalues can still be done for increasing real parts. The Gerschgorin's circle theorem [Gerschgorin, 1931; Bell, 1965] asserts that $\mathcal{L}$'s spectrum in the complex plane is fully contained within the union of disks $D(c_i, r_i)$ having as centers c_i the diagonal elements of $\mathcal{L}$ ($c_i = \mathcal{L}_{ii}$), and as radii r_i the sums of the absolute values of the other elements in the corresponding rows ($r_i = \sum_{i \neq j} |\mathcal{L}_{ij}|$)

$$\{\lambda_l\}_{l=1,\ldots,N} \subset \bigcup_i D\left(\mathcal{L}_{ii}, \sum_{j \neq i} \mid \mathcal{L}_{ij} \mid\right).$$

Mathematically, since $\mathcal{L}$ is a zero row-sum matrix, and

$$\mathcal{L}_{ii} = \sum_{j \neq i} |\mathcal{L}_{ij}|$$

because of the extra assumption that all nonzero off diagonal elements are negative, we can assume that *in all cases and for all network sizes* the $\mathcal{L}$'s spectrum is fully contained within the unit circle centered at 1 on the real axis ($|\lambda_l - 1| \leq 1$, $\forall\, l$), giving the following inequalities: (i) $0 < \lambda_2^r \leq \cdots \leq \lambda_N^r \leq 2$, and (ii) $\mid \lambda_l^i \mid \leq 1$, $\forall\, l$.

By calling $\mathcal{R}$ the bounded region in the complex plane where the master stability function $\Lambda(\nu)$ provides a negative Lyapunov exponent, the stability condition for the synchronous state is that the set $\{\sigma\lambda_l\}_{l=2,\ldots,N}$ be entirely contained in $\mathcal{R}$ for a given σ.

This is best accomplished for connection topologies that make both the ratio

$$\frac{\lambda_N^r}{\lambda_2^r}$$

and

$$\max_{l\geq 2}\{|\lambda_l^i|\}$$

as small as possible, to simultaneously avoid possible instabilities due to either the real or the imaginary part of some eigenvalues lying out of $\mathcal{R}$.

Master stability function arguments are currently used as a challenging framework for the study of synchronized behaviors in complex networks, especially for understanding the interplay between complexity in the overall topology and local dynamical properties of the coupled units.

Recent applications of the Master Stability Function for both symmetric and asymmetric coupling matrices [Motter *et al.*, 2005; Chavez *et al.*, 2005; Hwang *et al.*, 2005] have shown that suitably weighted network architectures are, indeed, able to greatly enhance the stability of the synchronization manifold, thus providing a direct way to investigate how the properties of the connection wiring influence the efficiency and robustness of the system.

3. Nonidentical Oscillators in Complex Networks: Synchronization, Entrainment and Selection of Topology

The above described formalism is a powerful tool for analyzing the stability of synchronous states in a network, but it explicitly requires that the units are identical to assure the existence of a complete synchronization manifold.

Real systems, however, hardly meet such a condition, as they are typically inhomogeneous. Consequently, any model that wants to furnish a reliable description of these situations, must necessarily include some degree of randomness.

As soon as nonidenticity in the networking elements is considered, however, there is no choice but to restrict oneself to numerical simulations on synchronization and control processes of complex networks [Moreno & Pacheco, 2004; McGraw & Menzinger, 2005; Leyva *et al.*, 2006].

In fact, some efforts have been made to extend the Master Stability Function formalism to the case of nonidentical oscillators. In particular, [Chavez *et al.*, 2005] numerically tested that, for a tiny parameter mismatch, the network's synchronous state has almost the equivalent stability properties predicted by a Master Stability Function approach applied to a network of identical oscillators for an average of the parameter distribution. Following this study, [Hramov *et al.*, 2008] later discussed how the Master Stability Function formalism can be extended to the case of a slightly dispersed distribution of parameters by noise addition. There, it was shown that, for small enough noise in the system, the calculation of the conditional largest Lyapunov exponents in Eq. (6) can be performed by considering as synchronization manifold the solution of a stochastic differential equation associated to the corresponding network of identical units.

Another scenario where the Master Stability Function approach has been extended is that of networks subject to pinning control [Sorrentino *et al.*, 2007]. There, the networking oscillators are identical, but they are further forced to follow a prescribed dynamics by a pinning process. The authors manage to define the network pinning controllability by studying the stability of the imposed dynamics in the $N+1$ phase space by the eigenratio λ_{N+1}/λ_2 calculated for a suitably extended connectivity matrix. The strategy to pin the network' nodes is either at random or it is selective following the node ranking given by some topological properties.

In this section, instead, we introduce a complimentary method to entrain a network of nonidentical oscillators, in which nodes are pinned depending on their instantaneous distances from a given reference dynamics [Sendiña-Nadal *et al.*, 2008]. The main difference is, therefore, that the pinning rule is here fully determined by the dynamical evolution of the network, and not at all influenced by topological properties of the system.

In order to substantiate this approach, we consider a network $\mathcal{G}_0$ made of N bidirectionally coupled Kuramoto phase oscillators [Kuramoto, 1984], in which each node is randomly connected to the rest with probability $p = \ln(N)/N$. On top of this structure, an extra phase oscillator with frequency ω_p, acting as a pacemaker, grows an evolving/adaptive network of unidirectional links to nodes in $\mathcal{G}_0$, as seen in Fig. 1.

Precisely, the process that we will describe below consists initially in selecting at time $t=0$ a specific initial network configuration $\mathcal{G}_0$. At subsequent times $t_k = k\Delta t$, the external pacemaker launches a new forcing connection to nodes in $\mathcal{G}_0$ (with a selecting rule that will be specified momentarily), allowing for multiple forcing connection to a single node.

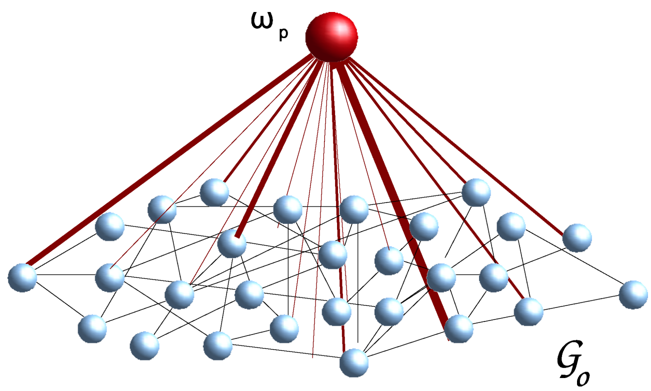

Fig. 1. Sketch of the growing process. The blue nodes and links represent a pristine random network $\mathcal{G}_0$, the red node is the external pacemaker at frequency ω_p, and the red links represent its unidirectional forcing connections with elements of $\mathcal{G}_0$. Furthermore, the thickness of every red connection is proportional to the weight associated with the forcing operated by the pacemaker on each specific element of $\mathcal{G}_0$.

The dynamics of the complete network is then given by the equation:

$$\dot{\phi}_i = \omega_{0i} + \frac{\varepsilon}{W_i(t)} \sum_{j=1}^{N} A_{ij} \sin(\phi_j - \phi_i) + \frac{\varepsilon_p}{W_i(t)} C_i(t) \sin(\phi_p - \phi_i), \quad i = 1, \ldots, N \tag{7}$$

ϕ_i are the phases of N oscillators in $\mathcal{G}_0$ (whose initial frequencies ω_{0i} are randomly chosen from a uniform distribution within the range 0.5 ± 0.25), and $\dot{\phi}_p = \omega_p$ is the frequency of the pacemaker.

ε and ε_p are suitable coupling parameters describing, respectively, the strengths of the bidirectional coupling among the elements of $\mathcal{G}_0$ and the unidirectional forcing operated by the pacemaker on the nodes of the pristine graph. In what follows, ε is selected so, at time $t = 0$ the pristine graph $\mathcal{G}_0$ features a unsynchronized motion, and the frequency of the pacemaker is set to be $\omega_p = 0.5$.

The coefficients A_{ij}, are the $N \times N$ elements of the adjacency matrix $\mathbf{A} = (A_{ij})$, describing the structure of the network of connections in $\mathcal{G}_0$ ($A_{ij} = 1$ if the oscillators i and j are connected and 0 otherwise). The coefficients $C_i(t)$ are the N components of a time dependent vector $\mathbf{C}$ accounting for the evolution in time of the connections of $\mathcal{G}_0$ with the pacemaker, with $C_i(t) = 0$ if the oscillator i is disconnected from the pacemaker at time t, and $C_i(t) = n_i \in \mathbb{N}$ when the oscillator i has received n_i forcing connections with the pacemakers during the time interval $[0, t]$. Notice that, in Eq. (7), these coupling strengths are properly scaled by the corresponding time dependent total link weight of each node in $\mathcal{G}_0$, which is given by:

$$W_i(t) = C_i(t) + \sum_{j=1}^{N} A_{ij} \tag{8}$$

As discussed above, the pinning process to entrain the network $\mathcal{G}_0$ to the pacemaker frequency is based on the dynamical state of the nodes rather than on their topological properties. Specifically, the selection criterion through which the pacemaker establishes and weights the links to $\mathcal{G}_0$ is as follows: at fixed time intervals Δt the pacemaker searches for that node in $\mathcal{G}_0$ whose instantaneous phase most accurately verifies an anti-phase condition with respect to its own.

In other words, the chosen node is which that holds more closely the phase condition:

$$\min_{i=1,\ldots,N} \{|\pi - \Delta\theta_i| \bmod 2\pi\}$$

where $\Delta\theta_i = \phi_i(t) - \phi_p(t)$. Then, the corresponding element of the pacemaker adjacency vector updates to $C_i(t + t_k) = C_i(t) + 1$.

Our main goal is to inspect the changes in the network topology induced by the entrainment process. The way the forcing process couples dynamics and topology leads, indeed, to some nontrivial features in the structure. As explained in Sec. 1, the appearance of a synchronized motion in a population of phase oscillators can be followed with the quantity introduced in Eq. (2).

To properly quantify the changes in the topology associated with the arousal of synchronization, we perform large trials of numerical simulations with $N = 1000$, $\varepsilon = 0.2$, and a pacemaker furnishing a total of 10 000 perturbations. We monitor the time evolution of the total incoming link weight distribution $P_t(W)$ of all nodes originally belonging to $\mathcal{G}_0$ during the process of forcing. It should be noted that the weighting of each link from the pacemaker to a given node is equivalent to a description in terms of having many identical pacemakers attached to that node with weight equal to one [Sendiña-Nadal *et al.*, 2008]. In this sense, the results obtained by monitoring the distribution of the weights can be translated to the corresponding in-degree distribution.

In fact, we here measure the *cumulative* link weight distribution $P_t^c(W)$, given by

$$P_t^c(W) = \sum_{W' > W} P_t(W').$$

This is because the summing process of $P(W)$ smoothens the statistical fluctuations generally present in the tail of the distribution. As a generic property, it is important to remark that, if a power-law scaling is observed in the behavior of $P^c(W)$ (i.e. if $P^c(W) \sim W^{-\gamma_c}$), this implies that also the degree distribution $P(W)$ is characterized by a power law scaling $P(W) \sim W^{-\gamma}$, with $\gamma \sim 1+\gamma_c$.

Figure 2 reports how $r(t)$ and $P_t^c(W)$ evolve in time in the two different coupling regimes: when ε_p is small enough so that the forcing does not lead to any entrainment [Figs. 2(a) and 2(b)], and for a process that eventually leads to entrainment of $\mathcal{G}_0$ to the frequency of the pacemaker [Figs. 2(c) and 2(d)].

One immediately realizes that in Fig. 2(b), $P_t^c(W)$ does not deviate significantly in shape from its initial distribution $P_0^c(W)$ (♦ symbols), and the only effect of the forcing on the weight distribution is to uniformly increase the mean weight. At the same time, the evolution of the order parameter shows [Fig. 2(a)] that the network is not able to reach a synchronous behavior.

At variance, Fig. 2(d) shows that the entrainment process (manifested by the evolution of $r(t)$ to 1 in Fig. 2(c)) is accompanied by the convergence of $P_t^c(W)$ to an asymptotic distribution $P_{fin}^c(W)$ (■ symbols) which features a power-law shape.

The difference in the final distributions for the nonentrained and entrained networks, and the convergence in this latter case of $P_t^c(W)$ to a scale free distribution $P_{fin}^c(W)$ is a generic property of Eq. (7) [Sendiña-Nadal *et al.*, 2008].

The mechanism that is responsible for the emergence of such a weight distribution, is intimately related to the way the external pacemaker locks the oscillators in $\mathcal{G}_0$ in the course of the time, and in particular, on the relationship between the probability for a given node of $\mathcal{G}_0$ to acquire connections during the growth of the forcing network and its initial frequency. Figure 3 reports the initial frequency of each node pinned by the pacemaker Figs. 3(a) and 3(b) for the two coupling regimes (nonentrainment, $\varepsilon_p = 0.2$ and final entrainment, $\varepsilon_p = 0.5$), as well as the corresponding histograms of the times a

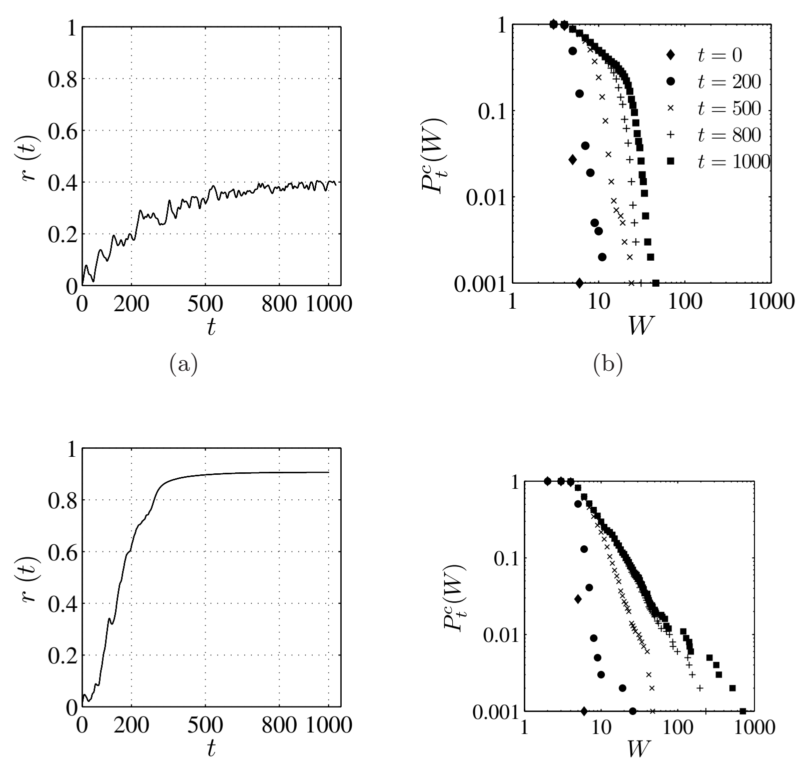

Fig. 2. Time evolution of the order parameter $r(t)$ and of the cumulative weight distribution $P^c(W)$ in a log–log plot, for a specific realization of the forcing process (parameters specified in the text). Panels (a) and (b) correspond to a nonentrained network ($\varepsilon_p = 0.2$) and panels (c) and (d) to a entrained one ($\varepsilon_p = 0.5$). The time instants at which the distributions are taken in (b) and (d) are indicated in (b). Notice that, in the entrained case, $P_t^c(W)$ converges to an asymptotic distribution (■) which features a power-law shape.

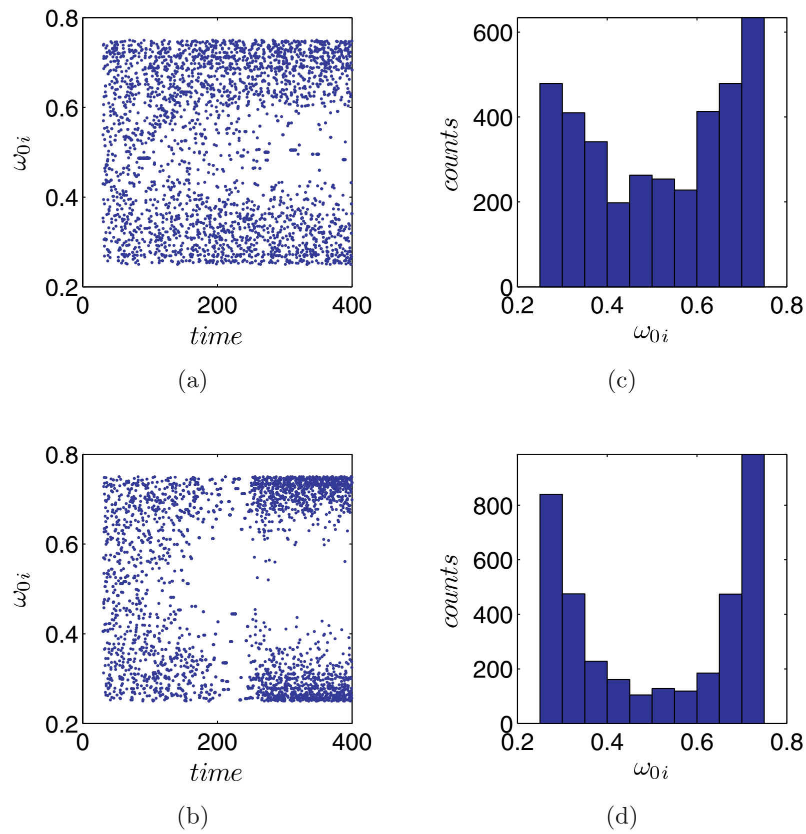

Fig. 3. (a) and (b) Initial frequency of the nodes pinned by the pacemaker as a function of time during the forcing period. (a) refers to an unsuccessful entrainment, and (b) to a successful one. (c) and (d) Number of times that a given initial frequency was chosen by the pacemaker for (c) an unsuccessful entrainment and (d) a successful one.

given initial frequency is hit by a connection from the pacemaker Figs. 3(c) and 3(d).

A close inspection of the figures makes it evident that for the nonentrainment regime, any initial frequency has almost the same probability of being selected by the pacemaker during the forcing process, whereas the successful entrainment case corresponds to a sort of preferential attachment selection which promotes more and more connections from the pacemaker to those nodes in $\mathcal{G}_0$ whose initial frequencies were more distant from that of the pacemaker. A full description and explanation of this mechanism is given in [Sendiña-Nadal *et al.*, 2008].

4. Conclusions

In conclusion, we have described some results on how topology and dynamics of a network of phase oscillators interplay during the settings of collective synchronized behavior. In particular, we have discussed how the stability of the synchronization dynamics is affected by the topology of the underlying wiring structure, and how dynamics and topology of a network can be controlled at once by means of a pinning mechanism which entrains the phases of the oscillators to that of an external pacemaker.

It has to be remarked that the very fact that a purely dynamical mechanism can induce the emergence of specific power-law degree distributions can provide new insights on the fundamental processes at the basis of the growth of some real world networks, that seem to ubiquitously feature such connectivity distributions.

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