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Integrability and symmetries for the Helmholtz oscillator with friction

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Abstract
This paper deals with the Helmholtz oscillator, which is a simple nonlinear oscillator whose equation presents a quadratic nonlinearity and the possibility of escape. When a periodic external force is introduced, the width of the stochastic layer, which is a region around the separatrix where orbits may exhibit transient chaos, is calculated. In the absence of friction and external force, it is well known that analytical solutions exist since it is completely integrable. When only friction is included, there is no analytical solution for all parameter values. However, by means of the Lie theory for differential equations we find a relation between parameters for which the oscillator is integrable. This is related to the fact that the system possesses a symmetry group and the corresponding symmetries are computed. Finally, the analytical explicit solutions are shown and related to the basins of attraction.

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1. Introduction

Oscillations and waves are ubiquitous in nature and are easily modelled with the help of differential equations. The general equation for the one-dimensional oscillatory motion of a unit mass particle can be easily understood using a mechanical analogy. Assume that the particle moves in a force field which is generated by a potential $V(x)$, then the general equation of motion may be written as

$$\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0. \tag{1}$$

Stating the problem this way, different oscillators may be obtained depending on the potential $V(x)$ acting on the particle. Assuming $V(x)$ to be a polynomial function in x, very few cases with analytical solutions have been studied. Among them are the Duffing oscillator, with a nonlinear term of fourth order, and the Helmholtz oscillator [1] where the cubic term is used. Obviously, higher order terms may be considered, which in general lead to rather complicated mathematical solutions. These are the nonlinear versions of the oscillator given by equation (1). If only the quadratic term is taken into account, obviously, the harmonic oscillator is derived. Another simple case with a non-polynomial potential $V(x) = -\cos x$ is the pendulum equation. It is important to remark that from all these four model equations complete analytical solutions are known in closed form. While circular functions are solutions of the harmonic oscillator, Jacobian elliptic functions are, in general, the solutions of the nonlinear oscillators considered here.

The study of an oscillator of this kind is worthwhile in spite of its apparent simplicity, because many physical problems of application may be reduced to the analysis of this simple nonlinear oscillator. The dynamics of the Helmholtz oscillator mimics the dynamics of certain pre-stressed structures, the capsizing of a ship [2] and the nonlinear dynamics of a drop in a time-periodic flow [3] or in a time-periodic electric field [4]. It appears also in relation to the randomization of solitary-like waves in boundary-layer flows [5] and in the three-wave interaction, also referred to as resonant triads [6].

If we include linear friction and a periodic force in equation (1), we obtain

$$\ddot{x} + \delta\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = \gamma\cos\omega t \tag{2}$$

where the inclusion of friction and force in the system bestows a rather different dynamical behaviour compared to the case without them.

Even though an analysis in the absence of friction has been accomplished for the pendulum equation [7, 8] and for the Duffing oscillator [9], no similar results are known for the Helmholtz oscillator. In spite of that, when friction is considered, this system has received some attention by different authors [1, 2, 10].

A way of studying the Helmholtz oscillator is by means of computational methods. Nowadays, the use of a computer allows the calculation of good approximations of the solution of many problems. However, analytical methods give important information about the dynamical behaviour of the system. Chaotic aspects of certain dynamical systems are better understood when the analytical structure is known [11]. Actually, the analytical structure comprises information about the integrability of the model, and this is useful to assure whether chaos is possible or not. This link between integrability and chaotic motion has been analysed for several models, for instance, the Lorenz model [12] or the Hénon–Heiles Hamiltonian [13].

The *Lie theory for differential equations* is a powerful method to analytically study a dynamical system. Actually, this theory was originally developed to study differential equations. Different techniques developed to solve certain types of equations (i.e., separable or exact equations) are regarded in this theory as special cases of a general integration method. The Lie theory allows the determination of the equation is integrable and its symmetry group. Basically, a symmetry group of a differential equation is a group which transforms solutions to other solutions of the equation. In the case of an ordinary differential equation, it is useful to integrate this since invariance under a symmetry implies that the order of the equation can be reduced by one. Hence for a second order equation, such as the Helmholtz oscillator, two symmetries are needed to integrate it and to write the solution in terms of known functions.

However, besides the exact formulae and expressions for a generic oscillator, it is important to remark that new insights and intuitions can be derived from its study, which may help understand the dynamics of other similar problems.

The organization of the paper is as follows. In section 2, the solution of the Helmholtz oscillator without friction inside the single-well potential is carried out. Also, the construction of the separatrix map and the expression of the stochastic layer width are shown. Section 3 shows when the Helmholtz oscillator with friction is integrable by means of the Lie theory of differential equations. The symmetries are calculated to obtain the general solution $x(t)$ and the first integral of motion is written and related to a Hamiltonian function. Also, the physical behaviour of this oscillator is related to the analytical solutions. Finally, section 4 provides the concluding remarks.

2. Dynamics of the Helmholtz oscillator

2.1. Introduction

The equation of motion of a particle of unit mass which undergoes a periodic force in a cubic single-well potential with friction, reads

$$\ddot{x} + \delta\dot{x} + \alpha x - \beta x^2 = \gamma \cos \omega t \tag{3}$$

where δ, α, β, γ and ω are positive constants.

A Hamiltonian and Lagrangian formalism can be used [14] to derive the equation of a particle in a potential $V(x)$ with a linear friction and a periodic force. The particular case given by equation (3) is derived from a time-dependent Hamiltonian and Lagrangian of the following form:

$$\begin{aligned} H(p, x, t) &= \tfrac{1}{2}p^2\, \mathrm{e}^{-\delta t} + \mathrm{e}^{\delta t} V(x, t) \\ L(\dot{x}, x, t) &= \mathrm{e}^{\delta t}\left[\tfrac{1}{2}\dot{x}^2 - V(x, t)\right] \end{aligned} \tag{4}$$

where $V(x, t)$ is the following generalized potential for the whole system:

$$V(x, t) = \frac{\alpha x^2}{2} - \frac{\beta x^3}{3} - \gamma x \cos \omega t. \tag{5}$$

In this section, it is considered that $\delta = 0$ (i.e., there is no friction). Hence, the equation to analyse is

$$\ddot{x} + \alpha x - \beta x^2 = \gamma \cos \omega t \tag{6}$$

and therefore, equation (4) becomes

$$\begin{aligned} H(p, x, t) &= \tfrac{1}{2}p^2 + V(x, t) \\ L(\dot{x}, x, t) &= \tfrac{1}{2}\dot{x}^2 - V(x, t) \end{aligned} \tag{7}$$

which will be particularly useful to compute the so-called separatrix map. This map yields a lot of information about the effect of a periodic force on the Helmholtz oscillator. In particular, about the possibility of transient chaos as a consequence of the force.

2.2. Single-well potential

When $\gamma = 0$, the equation of a conservative oscillator is obtained. This oscillator may be understood as a particle which is situated in a single potential well $V(x)$ defined as

$$V(x) = \frac{\alpha x^2}{2} - \frac{\beta x^3}{3}. \tag{8}$$

One important feature of this system, which can be easily seen in figure 1, is that according to the initial condition and the energy of the particle, the orbits may be bounded or unbounded. When the value of the energy $E_{\mathrm{min}} = 0 \leqslant E \leqslant E_{\mathrm{max}} = \frac{\alpha^3}{6\beta^2}$, then there exist possibilities of bounded motions and hence, oscillations, while for $E > E_{\mathrm{max}}$, the motion of the particle is unbounded, i.e., the particle escapes to infinity.

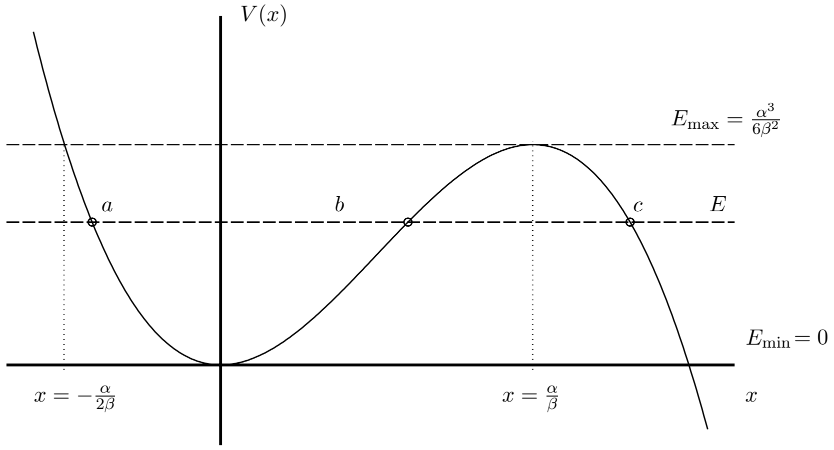

Figure 1. Potential energy associated with the Helmholtz oscillator, which may be seen as the simplest potential with an escape. Note that the potential has been chosen to be $V(x) = \frac{\alpha}{2}x^2 - \frac{\beta}{3}x^3$, because this way α and β are positive constants. The orbits will be bounded only when $-\frac{\alpha}{2\beta} < x < \frac{\alpha}{\beta}$ and $0 < E < E_{\max}$. For instance, the bounded orbit with energy E is comprised within $[a, b]$. If $x > c$, then the orbit is unbounded.

When the particle has energy E in the range $[E_{\min}, E_{\max}]$, then the cubic equation $E - V(x) = 0$ provides three real roots a, b and c $(a < b < c)$, which represent, physically, the *turning points*, i.e., the points where the velocity of the particle changes sign. These roots verifies the following relationships which will be important for further results:

$$a + b + c = \frac{3\alpha}{2\beta} \qquad ab + bc + ac = 0 \qquad abc = -\frac{3E}{\beta} \tag{9}$$

and their general expressions are

$$a = \frac{\alpha}{2\beta} + (-1 - m)\frac{\lambda}{3} \qquad b = \frac{\alpha}{2\beta} + (2m - 1)\frac{\lambda}{3} \qquad c = \frac{\alpha}{2\beta} + (2 - m)\frac{\lambda}{3} \tag{10}$$

where to obtain the former results, the following parameters are used:

$$m = \frac{b - a}{c - a} \qquad \lambda = c - a. \tag{11}$$

If it is also defined

$$\Delta^2 = 1 - m + m^2 \tag{12}$$

then from equations (9), it is derived that

$$\frac{\alpha}{2\beta} = \frac{\lambda\Delta}{3} \tag{13}$$

a useful expression which allows the values of the roots to be expressed in terms of only the parameter m

$$a = \frac{\alpha}{2\beta} + (-1 - m)\frac{\alpha}{2\beta\Delta} \qquad b = \frac{\alpha}{2\beta} + (2m - 1)\frac{\alpha}{2\beta\Delta} \qquad c = \frac{\alpha}{2\beta} + (2 - m)\frac{\alpha}{2\beta\Delta}. \tag{14}$$

2.3. General exact solution

Now, the equation of motion, equation (3), can be exactly solved in the conservative case, i.e., in the absence of friction and periodic force. Hence, the analytical solutions of the periodic orbits inside the single well will be derived.

The conservation of energy can be used to set the problem in terms of the three roots of $E - V(x) = 0$ in the following way:

$$\frac{\dot{x}^2}{2} = \frac{\beta}{3}(x-a)(x-b)(x-c). \tag{15}$$

The terms can be rearranged into

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2\beta}{3}}\sqrt{(x-a)(x-b)(x-c)} \tag{16}$$

and now, after a simple integration of the above equation, the following result is achieved:

$$t - t_0 = \sqrt{\frac{3}{2\beta}}\int_a^x \frac{\mathrm{d}x}{\sqrt{(x-a)(x-b)(x-c)}} \tag{17}$$

where it is assumed that the particle lies in $x = a$ for the initial time t_0. Now assume the following change of variable:

$$x = a + (b-a)\sin^2\theta \tag{18}$$

and introducing this result in equation (17) we obtain that

$$t - t_0 = \sqrt{\frac{6}{\beta(c-a)}}\int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{19}$$

The solution of the integral on the right-hand side is given by the *sine amplitude* of a Jacobian elliptic function [15] from where it is deduced that

$$\sqrt{\frac{\beta(c-a)}{6}}(t - t_0) = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = \mathrm{sn}^{-1}(\sin\phi; m) \tag{20}$$

where ϕ is the *elliptic amplitude* and m is the *elliptic parameter*. There is a lot of confusion in the literature about the use of the *elliptic parameter m* and the *elliptic modulus k*, which are related by the expression $k^2 = m$. The notation of [15] is followed here, where $\mathrm{sn}(u; k)$ represents the sine amplitude when the elliptic modulus is used, and $\mathrm{sn}(u; m)$ when the elliptic parameter is used. For simplicity, the elliptic parameter is used throughout.

Thus, from the last equation, it is inferred,

$$\sin\phi = \mathrm{sn}\left\{\sqrt{\frac{\beta(c-a)}{6}}(t - t_0); m\right\} \tag{21}$$

and if the change of variable used before is taken into account, the following solution is obtained:

$$x(t) = a + (b-a)\mathrm{sn}^2\left\{\sqrt{\frac{\beta(c-a)}{6}}(t - t_0); m\right\} \tag{22}$$

which is the general solution for all the periodic orbits lying within the single well. Note that all orbits are labelled by the elliptic parameter m. This parameter m which ranges from $0 \leqslant m \leqslant 1$ is in fact the same as previously defined in equation (11) in relation to the turning points of motion in the potential well. It labels the energy of each periodic orbit inside the potential well.

2.4. Period of the orbits

It is also interesting to calculate the period of each and every orbit inside the potential well. For this purpose the following integral has to be worked out:

$$T(m) = 2\sqrt{\frac{3}{2\beta}}\int_b^c \frac{\mathrm{d}x}{\sqrt{(x-a)(x-b)(x-c)}} = \sqrt{\frac{6}{\beta(c-a)}}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}. \tag{23}$$

The last integral represents exactly the *complete elliptic integral of the first kind* $K(m)$ [15], so that

$$T(m) = \sqrt{\frac{24}{\beta(c-a)}}K(m). \tag{24}$$

For orbits whose energy is very small in absolute terms, i.e., $m \to 0$, the complete elliptic integral of first kind $K(m) \to \frac{\pi}{2}$ and then the period becomes $T \to \sqrt{\frac{4}{\alpha}}\pi$. This is obviously the period for the linear oscillations around the elliptic fixed point (0, 0). However for values of the energy close to the separatrix, which means $m \to 1$, the complete elliptic integral of the first kind diverges logarithmically in this way,

$$K(m) \approx \frac{1}{2}\ln\left(\frac{16}{1-m}\right) \tag{25}$$

and this means that the period also diverges logarithmically for values of m close to unity

$$T(m) = \frac{2}{\sqrt{\alpha}}\ln\left(\frac{16}{1-m}\right). \tag{26}$$

2.5. Equation of the separatrix

From the general solution obtained before, it is rather easy to derive the equation of the separatrix orbit. In fact the separatrix orbit is the orbit with energy corresponding to the parameter $m = 1$ and which possesses a period infinity. The sine amplitude of the Jacobian elliptic function has two natural limiting functions depending on the limit values of m. These limiting functions are $\mathrm{sn}(u; m) \to \sin u$ for $m \to 0$ and $\mathrm{sn}(u; m) \to \tanh u$ for $m \to 1$.

Moreover, if $m = 1$ then $\Delta = 1$, $a = -\frac{\alpha}{2\beta}$ and $b = c = \frac{\alpha}{\beta}$ from equations (12) and (14). Hence, the equations in phase space are given by

$$\begin{aligned} x_{sx}(t) &= \frac{3\alpha}{2\beta}\left(\frac{2}{3} - \cosh^{-2}\left\{\sqrt{\frac{\alpha}{4}}(t-t_0)\right\}\right) \\ y_{sx}(t) &= \frac{3}{2}\sqrt{\frac{\alpha^3}{\beta^2}}\frac{\sinh\left\{\sqrt{\frac{\alpha}{4}}(t-t_0)\right\}}{\cosh^3\left\{\sqrt{\frac{\alpha}{4}}(t-t_0)\right\}} \end{aligned} \tag{27}$$

which has a fish-shaped form. Actually, it is easy to check that $y_{sx}(t)$ and $x_{sx}(t)$ are related as

$$y_{sx}^2 = \frac{2}{3}\beta\left(x_{sx} - \frac{\alpha}{\beta}\right)^2\left(x_{sx} + \frac{\alpha}{2\beta}\right). \tag{28}$$

The bounded motions lie in the interior of the separatrix, while the unbounded motions lie outside. In this case, the separatrix corresponds to a *homoclinic orbit* since the orbit connects the hyperbolic fixed point $\left(\frac{\alpha}{\beta}, 0\right)$ to itself.

2.6. Stochastic layer

Once the Helmholtz oscillator has been analysed, it is interesting to study how the orbits behave in the proximity of the separatrix when a periodic force is applied.

The time-dependent Hamiltonian in equation (7) can be used, as was explained in the introduction of this section, to study the Helmholtz oscillator with a periodic force. This time-dependent Hamiltonian can be seen as the sum of a time-independent Hamiltonian

$$H_0(x, p) = \frac{1}{2}p^2 + \frac{\alpha}{2}x^2 - \frac{\beta}{3}x^3 \tag{29}$$

and a time-dependent Hamiltonian

$$H_1(x, t) = -\gamma x \cos \omega t \tag{30}$$

i.e., the Hamiltonian $H(p, x, t)$ can be written as

$$H(p, x, t) = H_0(x, p) + H_1(x, t). \tag{31}$$

The former Hamiltonian allows the effect of the force to be analysed by means of an area preserving map, which is called the *whisker map* or the *separatrix map*. This map measures the energy and phase change of a trajectory close to the separatrix for each period of the motion [16].

In order to construct this map, the change of the Hamiltonian H_0 must be evaluated. The total derivative of H_0 is the following:

$$\frac{\mathrm{d}H_0}{\mathrm{d}t} = \{H_0, H\} = \{H_0, H_1\} = -\frac{\partial H_0}{\partial \dot{x}}\frac{\partial H_1}{\partial x} = \gamma \dot{x} \cos \omega t \tag{32}$$

where {} is the Poisson bracket.

Since our main interest is to discuss the motion of the particle when its energy is close to the separatrix, it is assumed that γ is small enough to consider that H_1 is a small perturbation. Then, it is close to the separatrix where big effects in the dynamics of the particle may be expected. The effect of a small perturbation on the orbits of small energy is negligible.

The method used to obtain the separatrix map, when H_1 is considered to be a small perturbation, is a standard one [16]. The first step is the computation of the energy ΔE. This energy accounts for the amount of the energy which an orbit close to the separatrix needs to accomplish a complete cycle and is given through the integration of equation (32),

$$\Delta E = \gamma \int_{\Delta t} \dot{x} \cos \omega t \, \mathrm{d}t \tag{33}$$

where $\Delta t = T/2 = \pi/\omega$. Note that this integral signals the border of the stochastic layer.

This energy is usually written in the following way to be evaluated around the separatrix:

$$\Delta E_n = \gamma \int_{t_n - \frac{T}{2}}^{t_n + \frac{T}{2}} \dot{x} \cos \omega t \, \mathrm{d}t \approx \gamma \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_{sx}}{\mathrm{d}t} \cos\{\omega(t + t_n)\} \, \mathrm{d}t. \tag{34}$$

From the third equality in equations (9) and (10), a relationship between the energy E and the parameter m is found. Expanding around $m = 1$ up to second order, the following expression is obtained: $8E \approx (1 - m)^2$. This approximation is used later to determine the separatrix map and its corresponding stochastic layer.

The change of the phase is given by $\Delta\phi = \omega T$. The expression for the energy relationship found before in terms of m, when m is close to 1, suggests that the period of the orbits close to the separatrix behaves like

$$T(m) \approx \frac{1}{\sqrt{\alpha}} \ln\left(\frac{32}{E}\right). \tag{35}$$

In this manner, the change of energy E and phase ϕ from the period n to the period $n+1$ is given by the separatrix mapping [8]

$$E_{n+1} = E_n + \Delta E_n \qquad \phi_{n+1} = \phi_n + \omega T_{n+1} \tag{36}$$

where the variables (E, ϕ) are to be understood as a canonical pair. This map contains in principle the essential dynamics in the region close to the separatrix. Thus, the separatrix map is given by

$$E_{n+1} = E_n + \frac{6\pi\omega^2}{\beta}\frac{\gamma\sin\phi_n}{\sinh\left\{\frac{\pi\omega}{\sqrt{\alpha}}\right\}} \qquad \phi_{n+1} = \phi_n + \frac{\omega}{\sqrt{\alpha}}\ln\left(\frac{32}{E_{n+1}}\right). \tag{37}$$

Another way of measuring the instability is through the calculation of the following parameter K, defined as [8]

$$K = \left|\frac{\delta\phi_{n+1}}{\delta\phi_n} - 1\right| \tag{38}$$

from which the stochastic layer width is achieved as a by-product. It supplies the information about how a small phase interval is stretched. The measure of the local instability is given by $K \geqslant 1$, because close to the separatrix, a small change in frequency may cause a big effect in phase. The stochastic layer width is given by the value

$$E \approx \frac{6\pi\gamma\omega^3}{\sqrt{\alpha}\beta\sinh\left\{\frac{\pi\omega}{\sqrt{\alpha}}\right\}} \tag{39}$$

which corresponds to the width of the region close to the separatrix where it is likely to expect chaotic motions.

3. Dynamics of the Helmholtz oscillator with friction

3.1. Introduction

In this section, the Helmholtz oscillator in equation (3) is analysed in the absence of the periodic force, i.e., when $\gamma = 0$. Then, the equation of motion of a particle of unit mass reads

$$\ddot{x} + \delta\dot{x} + \alpha x - \beta x^2 = 0. \tag{40}$$

To investigate the integrability of this equation, the *Lie theory of differential equations* will be used [17, 18]. However, it should be noted that the integrability of a differential equation can also be analysed by means of Kowalewski's asymptotic method (also called the Painlevé singularity structure analysis) and the same result is achieved. For example, in [19, 20] the Duffing oscillator is analysed in this manner. Nevertheless, the Lie theory is used in this work because this approach, in addition to giving information about when the equation is integrable, allows the problem to be reduced to canonical variables which eases the integration of the equation in a more general and natural way.

It can be seen in [17, 18] that in order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \eta(t,x)\frac{\partial}{\partial x} + \xi(t,x)\frac{\partial}{\partial t} \tag{41}$$

it is needed to find an infinitesimal operator X_{+2} such that

$$X_{+2}(\ddot{x} + \delta\dot{x} + \alpha x - \beta x^2) = 0. \tag{42}$$

The operator X_{+2} is

$$X_{+2} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + A(t,x,\dot{x})\frac{\partial}{\partial \dot{x}} + B(t,x,\dot{x},\ddot{x})\frac{\partial}{\partial \ddot{x}} \tag{43}$$

where $A(t,x,\dot{x})$ and $B(t,x,\dot{x},\ddot{x})$ are defined as follows:

$$\begin{aligned} A(t,x,\dot{x}) &= \eta_t + \dot{x}(\eta_x - \xi_t) - \dot{x}^2\xi_x \\ B(t,x,\dot{x},\ddot{x}) &= \eta_{tt} + \dot{x}(2\eta_{xt} - \xi_{tt}) + \dot{x}^2(\eta_{xx} - 2\xi_{tx}) - \dot{x}^3\xi_{xx} + \ddot{x}(\eta_x - 2\xi_t - 3\dot{x}\xi_x) \end{aligned} \tag{44}$$

with the usual notation $\omega_z \equiv \frac{\partial\omega}{\partial z}$.

All $\xi(t,x)$ and $\eta(t,x)$ that verify equation (42) generate infinitesimal operators X as in equation (41) which comprise the symmetries of the differential equation. Also, it is known that one symmetry can be used to reduce by one the order of a differential equation. Thus, to integrate a second order differential equation two symmetries are needed. Hence, the Helmholtz oscillator will be integrated only if $\xi(t,x)$ and $\eta(t,x)$ are such that they generate two linearly independent infinitesimal operators.

3.2. *Condition of integrability*

Following the procedure to determine the symmetries of a differential equation mentioned in the former section, equation (42) reads

$$\begin{aligned} X_{+2}(\ddot{x} + \delta\dot{x} + \alpha x - \beta x^2) &= \eta(\alpha - 2\beta x) + \delta(\eta_t + \dot{x}(\eta_x - \xi_t) - \dot{x}^2\xi_x) + \eta_{tt} + \dot{x}(2\eta_{xt} - \xi_{tt}) \\ &\quad + \dot{x}^2(\eta_{xx} - 2\xi_{xt}) - \dot{x}^3\xi_{xx} - (\delta\dot{x} + \alpha x - \beta x^2)(\eta_x - 2\xi_t - 3\dot{x}\xi_x). \end{aligned} \tag{45}$$

This is a polynomial of third degree in $[\dot{x}]$ which is zero if and only if the coefficients of every monomial is zero

$$[\dot{x}^3]\colon \xi_{xx} = 0 \tag{46}$$

$$[\dot{x}^2]\colon \eta_{xx} - 2\xi_{xt} + 2\delta\xi_x = 0 \tag{47}$$

$$[\dot{x}]\colon 2\eta_{xt} - \xi_{tt} + 3\xi_x(\alpha x - \beta x^2) + \delta\xi_t = 0 \tag{48}$$

$$[1]\colon \eta(\alpha - 2\beta x) + \delta\eta_t + \eta_{tt} - (\eta_x - 2\xi_t)(\alpha x - \beta x^2) = 0. \tag{49}$$

From the condition in equation (46), it is obvious that $\xi(x,t) = f(t) + k(t)x$, and this result in equation (47) implies that $\eta(x,t) = (k'(t) - \delta k(t))x^2 + xg(t) + h(t)$. If both results are used in equation (48) it is deduced that

$$4(k'' - \delta k')x + 2g' - (f'' + k''x) + 3k(\alpha x - \beta x^2) + \delta(f' + k'x) = 0. \tag{50}$$

This is a polynomial of second degree in $[x]$ which is zero if and only if the following three equations are verified:

$$\begin{aligned} &[x^2]\colon 3\beta k = 0 \\ &[x]\colon k'' + 3\delta k' - 3\alpha k = 0 \\ &[1]\colon 4k'' - f'' + \delta f' + 2g' = 0. \end{aligned} \tag{51}$$

These three equations imply that $k = 0$, hence $\xi(x,t) = f(t)$ and $\eta(x,t) = xg(t) + h(t)$, with the following relation between $f(t)$ and $g(t)$:

$$\delta f' + 2g' - f'' = 0. \tag{52}$$

According to these results, the condition in equation (49) is reduced to

$$(gx + h)(\alpha - 2\beta x) + \delta(xg' + h') + xg'' + h'' + (\alpha x - \beta x^2)(-g + 2f') = 0. \tag{53}$$

This is a polynomial of second degree in $[x]$ which is zero if and only if the following three equations are verified:

$$[x^2]\colon g + 2f' = 0 \tag{54}$$

$$[x]\colon 2\alpha f' + \delta g' + g'' - 2\beta h = 0 \tag{55}$$

$$[1]\colon \alpha h + \delta h' + h'' = 0. \tag{56}$$

The conditions in equations (52) and (54) imply that $g = A\,\mathrm{e}^{\frac{1}{5}\delta t}$, where A is a constant. When this result is used in equation (55) it is obtained that $h = \frac{1}{2\beta}\left(\frac{6}{25}\delta^2 - \alpha\right)g$. And finally, this result in equation (56) means that $\frac{1}{2\beta}\left(\frac{6}{25}\delta^2 + \alpha\right)\left(\frac{6}{25}\delta^2 - \alpha\right)g = 0$. But since it is supposed that $\alpha > 0$ and so $\frac{6}{25}\delta^2 + \alpha > 0$, there are only two options to verify all conditions.

The first one is when $g = 0$. In this case $h = 0$ and $f =$ constant, and this means that $\eta = 0$ and $\xi =$ constant. Hence, only one infinitesimal operator is obtained, namely $X = \partial_t$, and as a consequence the differential equation is partially integrable.

The second option in order to get two symmetries is when

$$\alpha = \frac{6}{25}\delta^2. \tag{57}$$

In this case $h = 0$ and $g = A\,\mathrm{e}^{\frac{1}{5}\delta t}$, which implies that $f = B - \frac{5}{2\delta}A\,\mathrm{e}^{\frac{1}{5}\delta t}$ and consequently $\xi = B - \frac{5}{2\delta}A\,\mathrm{e}^{\frac{1}{5}\delta t}$ and $\eta = Ax\,\mathrm{e}^{\frac{\delta}{5}t}$. Therefore two infinitesimal generators are found, namely,

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = -\frac{5}{2\delta}\,\mathrm{e}^{\frac{1}{5}\delta t}\frac{\partial}{\partial t} + x\,\mathrm{e}^{\frac{1}{5}\delta t}\frac{\partial}{\partial x}. \tag{58}$$

In conclusion, only when it is verified that $\alpha = \frac{6}{25}\delta^2$, the Helmholtz oscillator with friction is completely integrable. Therefore, there is a lot of information about the oscillator in this particular case, but it should be noted that the information applies for just a two-dimensional (2D) manifold in the parameter space $\{\delta, \alpha, \beta, \gamma\}$. When $\alpha \neq \frac{6}{25}\delta^2$, the oscillator is only partially integrable and there is no way to write down the solution in terms of known functions.

3.3. Reduction to canonical variables

The infinitesimal generators X_1 and X_2 defined in equations (58) are a 2D algebra L_2 since $[X_1, X_2] = \frac{\delta}{5}X_2$, where [ ] is a *commutator* called a Lie bracket, defined in the following manner: $[X_1, X_2] = X_1X_2 - X_2X_1$. This *Lie algebra* can be classified according to its structural properties [17] as type III because $[X_1, X_2] = \frac{\delta}{5}X_2 \neq 0$ and $X_1 \vee X_2 = x\,\mathrm{e}^{\frac{1}{5}\delta t} \neq 0$, where $\vee$ is a *pseudoscalar product* defined as $X_1 \vee X_2 = \xi_1\eta_2 - \xi_2\eta_1$, if $X_i = \xi_i\partial_1 + \eta_i\partial_2$ for $i = 1, 2$. Actually, L_2 is the algebra of the homothety transformations of the real line $\mathbb{R}$, where X_1 is a homothety operator and X_2 is a translation operator.

Then, it is known that there exists a pair of variables w and z, called *canonical variables*, which linearizes the action of the group G on $\mathbb{R}$ and reduces the algebra L_2 to $X_1 = w\partial_w + z\partial_z$ and $X_2 = \partial_z$.

Let w and z be

$$w \equiv Ax\,\mathrm{e}^{\frac{2}{5}\delta t} \qquad z \equiv B\,\mathrm{e}^{-\frac{1}{5}\delta t} \tag{59}$$

where A and B are constants, then

$$X_1 = \frac{2\delta}{5}\omega\frac{\partial}{\partial\omega} - \frac{\delta}{5}z\frac{\partial}{\partial z} \qquad X_2 = \frac{B}{2}\frac{\partial}{\partial z}. \tag{60}$$

Although it is not the canonical form, there is no need to introduce more changes because it is simple enough to reduce the Helmholtz oscillator to an easily integrable equation.

From the definitions stated in equations (59) the following result is obtained:

$$\begin{aligned} w'' &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) = \frac{25A}{B^2\delta^2}\,\mathrm{e}^{\frac{1}{5}\delta t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\dot{x}+\frac{2}{5}\delta x\right)\mathrm{e}^{\frac{3}{5}\delta t}\right) \\ &= \frac{25A}{B^2\delta^2}\,\mathrm{e}^{\frac{4}{5}\delta t}\left(\ddot{x}+\delta\dot{x}+\frac{6\delta^2}{25}x\right) = \frac{25\beta}{\delta^2 AB^2}w^2. \end{aligned} \tag{61}$$

Therefore, if A and B are chosen such that

$$AB^2 = \frac{25\beta}{6\delta^2} \tag{62}$$

then $w'' = 6w^2$, which is easily integrated yielding

$$(w')^2 = 4w^3 - g_3 \tag{63}$$

where g_3 is a constant.

The solution of this differential equation is the *Weierstrass function* $\wp(z; 0, g_3)$, since $\wp(z; g_2, g_3)$ verifies that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. Hence, the solution of the Helmholtz oscillator with friction is $w = \wp(z; 0, g_3)$, which is called the *equianharmonic case of the Weierstrass function* because $g_2 = 0$ [15].

It should be noted that $g_3 = 4w^3 - (w')^2$ is a first integral of motion and when a change of variables from (w, z) to (x, t) is carried out in equation (63), the first integral g_3 becomes $I(t, x, \dot{x})$ in this manner,

$$\left[\left(\dot{x}+\tfrac{2}{5}\delta x\right)^2 - \tfrac{2}{3}\beta x^3\right]\mathrm{e}^{\frac{6}{5}\delta t} = \Lambda g_3 = I(t, x, \dot{x}) \tag{64}$$

where $\Lambda = \left(\frac{6B^3\delta^3}{125\beta}\right)^2$ and, consequently, is always a positive constant.

The former result is an explicitly time-dependent first integral which is analogous to the first integral of the Duffing oscillator obtained in [20]. Also, it can be related to the Hamiltonian function of the Helmholtz oscillator with friction in the following way. Define two variables p and q as follows:

$$p = \sqrt{2}\left(\dot{x}+\tfrac{2}{5}\delta x\right)\mathrm{e}^{\frac{3}{5}\delta t} \qquad q = \sqrt{2}x\,\mathrm{e}^{\frac{2}{5}\delta t} \tag{65}$$

so the first integral $I(t, x, \dot{x})$ can be written as

$$I(p, q) = \frac{1}{2}p^2 - \frac{\beta}{3\sqrt{2}}q^3. \tag{66}$$

Define a function $H(p, q, t)$ related to the first integral $I(p, q)$ as

$$H(p, q, t) = I(p, q)\,\mathrm{e}^{-\frac{1}{5}\delta t} = \left(\frac{1}{2}p^2 - \frac{\beta}{3\sqrt{2}}q^3\right)\mathrm{e}^{-\frac{1}{5}\delta t}. \tag{67}$$

This function verifies the Hamilton equations, namely,

$$\begin{aligned} \frac{\partial H}{\partial p} &= p\,\mathrm{e}^{-\frac{1}{5}\delta t} = \sqrt{2}\left(\dot{x}+\frac{2}{5}\delta x\right)\mathrm{e}^{\frac{2}{5}\delta t} = \dot{q} \\ \frac{\partial H}{\partial q} &= -\frac{\beta}{\sqrt{2}}q^2\,\mathrm{e}^{-\frac{1}{5}\delta t} = -\sqrt{2}\beta x^2\,\mathrm{e}^{\frac{3}{5}\delta t} = -\dot{p} \end{aligned} \tag{68}$$

and hence $H(p, q, t)$ is a Hamiltonian function. Moreover, by means of equations (68) it is obtained that

$$\ddot{q} = \left(\dot{p} - \frac{1}{5}\delta p\right)\mathrm{e}^{-\frac{1}{5}\delta t} = \frac{\beta}{\sqrt{2}}q\,\mathrm{e}^{-\frac{2}{5}\delta t} - \frac{1}{5}\delta p\,\mathrm{e}^{-\frac{1}{5}\delta t} \tag{69}$$

which can be written in terms of (x, t) by using equations (65) as

$$\sqrt{2}\,\mathrm{e}^{\frac{2}{5}\delta t}\left(\ddot{x}+\delta\dot{x}+\frac{6\delta^2}{25}x-\beta x^2\right)=0. \tag{70}$$

Therefore, $H(p, q, t)$ is the Hamiltonian function of the Helmholtz oscillator with friction for the integrable case since the solutions to $\ddot{x}+\delta\dot{x}+\frac{6\delta^2}{25}x-\beta x^2=0$ and the solutions to the Hamilton equations of $H(p, q, t)$ are the same. Then, two remarks can be made. First, the explicitly time-dependent Hamiltonian is not a first integral of motion, which is reasonable since the energy is not constant in this system because of the friction. Second, the first integral $I(p, q)$ can be seen as the energy of a particle in a potential $V(q)=-\frac{\beta}{3\sqrt{2}}q^3$ and, thus, the Helmholtz oscillator can be regarded as a system with energy $I(p, q)$ at $t=0$, which vanishes exponentially with time.

3.4. Solutions of the integrable case

3.4.1. Case $g_3 = 0$. The equation to solve is $(w')^2 = 4w^3$, whose solution is $w=(z-c')^{-2}$ where c' is an arbitrary constant. The definitions of w and z and the relation in equation (62) imply that

$$x(t)=\frac{6\delta^2}{25\beta}\left(1+c_2\,\mathrm{e}^{\frac{1}{5}\delta t}\right)^{-2} \tag{71}$$

where c_2 is an arbitrary constant because c' is arbitrary.

3.4.2. Case $g_3 > 0$. The Weierstrass function $\wp(z; g_2, g_3)$ for $g_2 = 0$ and $g_3 > 0$ can be written in terms of the Jacobian elliptic cosine cn [15] as

$$w(z)=r+H\frac{1+\mathrm{cn}(2\sqrt{H}z+c_2;m)}{1-\mathrm{cn}(2\sqrt{H}z+c_2;m)} \tag{72}$$

with c_2 an arbitrary constant and where $m=\frac{2-\sqrt{3}}{4}\simeq 0.067$ and $H=\sqrt{3}r$ with $r=\sqrt[3]{\frac{g_3}{4}}$. Note that, as was explained in section 2.3., the elliptic parameter m is being used instead of the elliptic modulus k, which are related as $k^2=m$.

By using the definitions of w and z and the relation in equation (62), the following result, in terms of t, is obtained:

$$x(t)=\frac{6\delta^2}{100\beta}c_1^2\left(\frac{1}{\sqrt{3}}+\frac{1+\mathrm{cn}\left(c_1\,\mathrm{e}^{-\frac{1}{5}\delta t}+c_2;m\right)}{1-\mathrm{cn}\left(c_1\,\mathrm{e}^{-\frac{1}{5}\delta t}+c_2;m\right)}\right)\mathrm{e}^{-\frac{2}{5}\delta t} \tag{73}$$

where $c_1 = 2\sqrt{H}B$ and hence c_1 is arbitrary because B is arbitrary.

3.4.3. Case $g_3 < 0$. It is known [15] that $\wp(z; g_2, g_3) = -\wp(\mathrm{i}z; g_2, -g_3)$. This relation lets us apply the result in equation (72) for $g_3 < 0$ in the following way:

$$w(z)=-r'-H'\frac{1+\mathrm{cn}(2\sqrt{H'}\,\mathrm{i}z+\mathrm{i}c_2;m)}{1-\mathrm{cn}(2\sqrt{H'}\,\mathrm{i}z+\mathrm{i}c_2;m)} \tag{74}$$

where $m=\frac{2-\sqrt{3}}{4}$ and $H'=\sqrt{3}r'$ with $r'=\sqrt[3]{\frac{|g_3|}{4}}$. By means of the relation $\mathrm{cn}(\mathrm{i}u;m)\,\mathrm{cn}(u;m')=1$ where $m+m'=1$, it is possible to write equation (74) as

$$w(z)=-r'+H'\frac{1+\mathrm{cn}(2\sqrt{H'}z+c_2;m')}{1-\mathrm{cn}(2\sqrt{H'}z+c_2;m')}. \tag{75}$$

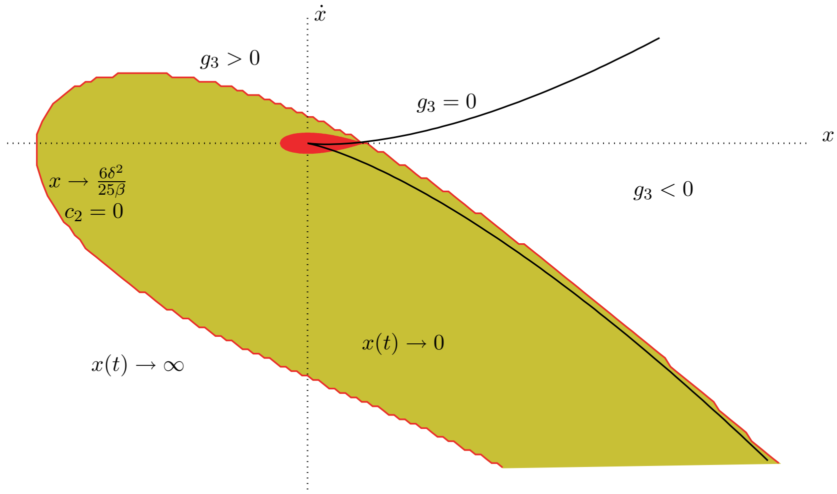

Figure 2. Relation between the geometry of the basins of attraction and the analytical features of the exact solutions when the Helmholtz oscillator is integrable. The grey region represents the initial conditions which tend to (0, 0) and the white region represents the ones tending to infinity. The boundary between both basins corresponds to the set of initial conditions tending to the local maximum and whose solutions have $c_2 = 0$. Also, the curve $g_3 = 0$ is depicted and represents the initial conditions whose solutions have the first integral of motion $I(t, x, \dot{x}) = 0$. Finally, the region where there are bounded orbits in the absence of friction is shown in dark grey. It is comparatively smaller than the region $x \to 0$ because the integrable case implies a large friction since $\alpha = \frac{6}{25}\delta^2$ and, hence, dissipation makes more initial conditions end up inside the potential well.

Hence, the solution may be written in terms of t by changing variables and using equation (62),

$$x(t) = \frac{6\delta^2}{100\beta} c_1^2 \left(-\frac{1}{\sqrt{3}} + \frac{1 + \mathrm{cn}\left(c_1\, \mathrm{e}^{-\frac{1}{5}\delta t} + c_2;\, m'\right)}{1 - \mathrm{cn}\left(c_1\, \mathrm{e}^{-\frac{1}{5}\delta t} + c_2;\, m'\right)} \right) \mathrm{e}^{-\frac{2}{5}\delta t} \tag{76}$$

where $m' = \frac{2+\sqrt{3}}{4} \simeq 0.933$ and $c_1 = 2\sqrt{H'}B$ and hence c_1 is arbitrary because B is arbitrary.

3.4.4. Discussion. In figure 2, the two basins of attraction of the Helmholtz oscillator are depicted in the phase space. The grey region represents the set of initial conditions which end up in the attractor (0, 0). It corresponds to bounded orbits in the phase space which asymptotically spiral inside the potential well. The white region is the set of initial conditions which correspond to unbounded orbits, i.e., tending to infinity. The boundary between both sets is formed by the stable manifold of an unstable periodic orbit. Actually, this orbit is the one that stays forever on the local maximum $\left(\frac{6\delta^2}{25\beta}, 0\right)$ of the potential, which means that all points in the boundary tend asymptotically to this point.

The basins of attraction are related to the analytical solutions via c_2 and to check this, it is necessary to study the asymptotical behaviour of the solutions. To calculate the limit $t \to \infty$ when $g_3 > 0$, the following change of variable $z \equiv c_1\, \mathrm{e}^{-\frac{1}{5}\delta t}$ is carried out, so the former limit becomes $z \to 0$. This implies in equation (72) that

$$\lim_{t\to\infty} x(t) = \lim_{z\to 0} \frac{6\delta^2}{100\beta} \left(\frac{1}{\sqrt{3}} + \frac{1 + \mathrm{cn}(z + c_2;\, m)}{1 - \mathrm{cn}(z + c_2;\, m)} \right) z^2. \tag{77}$$

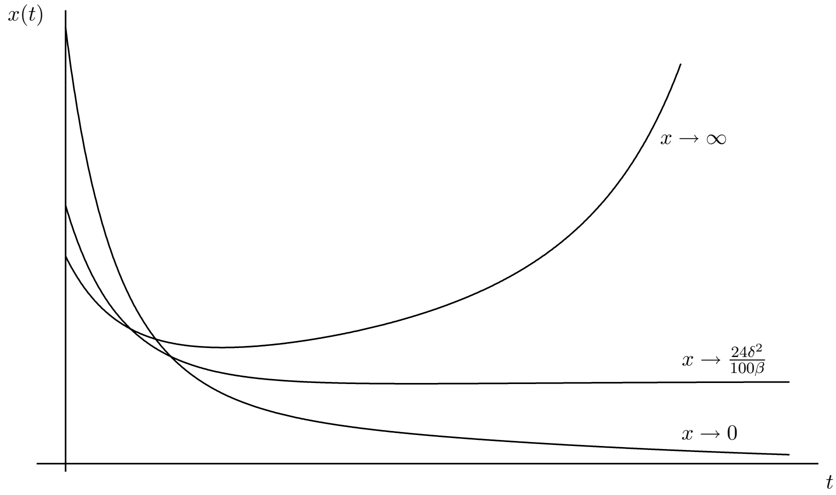

Figure 3. The phase space of the Helmholtz oscillator with friction has two basins of attraction and hence there are three kinds of orbits. Orbits spiralling inside the potential well tending to the minimum $x \to 0$, orbits tending to infinity $x \to \infty$ and orbits tending to the local maximum $x \to \frac{24\delta^2}{100\beta}$, which correspond to initial conditions upon the boundary of both basins. Note that particles are so damped in the integrable case that inside the potential well they go straight to zero instead of spiralling and so there are no oscillations in the curve $x \to 0$.

It should be noted that the Jacobian elliptic function $\mathrm{cn}(z; m)$ is a periodic function since $\mathrm{cn}(z+2K; m) = -\mathrm{cn}(z; m)$, i.e., $2K$ plays a role similar to π in a circular function. In fact, cn is periodic with period $4K$ where $2K \simeq 3.197$ because $m = \frac{2-\sqrt{3}}{4}$ and, thus, c_2 is comprised within $(-2K, 2K)$. Consequently, if $c_2 = 4NK$ with $N \in \mathbb{Z}$, then

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{z\to 0} \frac{6\delta^2}{100\beta}\left(\frac{1}{\sqrt{3}} + \frac{1+\mathrm{cn}(z; m)}{1-\mathrm{cn}(z; m)}\right) z^2 \\ &= \lim_{z\to 0} \frac{6\delta^2}{100\beta}\left(\frac{1}{\sqrt{3}} + \frac{4-z^2}{z^2}\right) z^2 = \frac{6\delta^2}{25\beta}\end{aligned} \tag{78}$$

where the following result has been used: $\mathrm{cn}(z; m) = 1 - \frac{1}{2}z^2 + o(z^4)$ [15]. Therefore, the boundary when $g_3 > 0$ can be defined as the points in the phase space whose analytical solutions have $c_2 = 0$.

When $g_3 < 0$, the result $x(t \to \infty) = \frac{6\delta^2}{25\beta}$, is also achieved when $c_2 = 0$. However, now $\mathrm{cn}(z; m')$ is a periodic function with $2K' \simeq 5.535$ since $m' = \frac{2+\sqrt{3}}{4}$, and thus c_2 is comprised within $(-2K', 2K')$. Nevertheless, in this case also the boundary can be defined as the points in phase space whose analytical solutions have $c_2 = 0$. Also, it is easy to verify from equation (71) that in the case $g_3 = 0$ the solution tends to $\frac{6\delta^2}{25\beta}$ when $c_2 = 0$.

In summary, the condition $c_2 = 0$ on the exact solutions yields the boundary between the two basins of attraction, which links the geometry of these two regions in the phase space with an analytical feature in the exact solutions.

Inside the grey region in figure 2, the region where there are bounded orbits in the absence of friction can be seen in black. It is a small region compared to the integrable case because $\alpha = \frac{6\delta^2}{25}$ and then dissipation is more important than its potential energy. In other words, many initial conditions which were unbounded orbits without friction dissipate energy quickly in this case and, as they go by the potential well, are trapped in it.

The existence of a strong dissipation in the integrable case also explains why there is no oscillatory behaviour in figure 3. When the orbit tends to the minimum inside the well, the particle is so damped that it goes straight to that minimum.

4. Concluding remarks

The Helmholtz oscillator is a simple model for studying phenomena which under certain conditions present a stable behaviour of oscillatory kind, but for other conditions, the behaviour is unstable (i.e., this oscillator presents an escape). Then, a question of interest is what happens close to the separatrix when a forcing term is introduced. The effect of force is not relevant for an orbit with little energy (i.e., close to the minimum in the potential well), because essentially its stable behaviour is not altered by the force. The width of the stochastic layer by using the separatrix map has been computed here. This gives the width of the energy band around the separatrix where it is likely that an orbit presents transient chaos.

An important aspect considered in this paper is the inclusion of friction. To solve the equation of the Helmholtz oscillator with friction and without forcing the Lie theory for differential equations is used. We show that the Helmholtz oscillator is completely integrable only when a certain relation between the parameters is satisfied. When this relation is not satisfied, the equation is partially integrable. Also, we calculate that the symmetries for the completely integrable case are a translation and a homothopy. Moreover, these two symmetries are the 2D algebra of the homothety transformations of the real line and the symmetry for the partially integrable case is a translation.

A first integral of motion is obtained when the equation is integrated by using one symmetry. We prove that this time-dependent integral of motion is related to a Hamiltonian function. The second symmetry allows integrating the first integral of motion to obtain, as a solution, the Weierstrass function. Finally, we write this solution in terms of Jacobian elliptic functions to show that there exists a relation with the basins of attraction of the oscillator.

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