Inter-layer synchronization in multiplex networks of identical layers

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Inter-layer synchronization is a distinctive process of multiplex networks whereby each node in a given layer evolves synchronously with all its replicas in other layers, irrespective of whether or not it is synchronized with the other units of the same layer. We analytically derive the necessary conditions for the existence and stability of such a state, and verify numerically the analytical predictions in several cases where such a state emerges. We further inspect its robustness against a progressive de-multiplexing of the network, and provide experimental evidence by means of multiplexes of nonlinear electronic circuits affected by intrinsic noise and parameter mismatch. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4952967]

In the last few years, partly motivated by the availability of ever larger and more detailed datasets, the study of real complex systems is benefiting from approaches based on the representation of such systems as networks of several layers interrelated between them. When the layers are composed of the same nodes and the only inter-layer interactions are those between the same nodes in the different layers, the multilayer structure is called a multiplex. Such a representation helps to understand, for instance, the spreading of an epidemic process due to social interactions occurring at different levels, like the physical and online levels. In this work we focus on the emergence of collective dynamical effects in multiplex networks, specifically on the as yet *unnoticed* phenomenon of inter-layer synchronization, whereby each constituent in a given layer of a system undergoes a synchronous evolution with all its replicas in other layers, *regardless* of whether or not it is synchronized with the other units of the same layer. In particular, we derive the conditions for the existence and stability of such a solution and inspect its robustness by means of numerical simulations and experiments with multiplexes of nonlinear electronic circuits. Our findings provide novel hints that may be useful in elucidating fundamental questions in regard to emerging phenomena in complex systems, such as how biological systems can collectively organize in a redundant way so that their effective functioning occurs through distinct (yet synchronized) layers of interactions.

I. INTRODUCTION

Synchronization in networked systems is one of the hottest topics of current research in nonlinear science.[1,2] So far, most of the focus has been concentrated on systems where all the constituents are treated on an equivalent footing, while only in the last few years the interest has moved towards incorporating the *multilayer* character of real world networks, by representing them as graphs formed by diverse layers,[3,4] which may either coexist or alternate in time.[4] For instance, epidemic processes need a multilayer representation to be properly described,[5] and some of the classical examples of pattern formation (like those in Refs. 6 and 7) find a suitable description within such a formalism.[8,9] Moreover, the consideration of a multilayer structure leads to interesting novel phenomena concerning the time scales of diffusion-like processes.[10] As far as dynamical processes are concerned, the multilayer formulation allows identifying synchronization regions that arise as a consequence of the interplay between the layers' topologies,[11,12] assessing the stability of a global synchronous state in a network of oscillators coupled through different variables,[13] as well as defining new types of synchronization based on the coordination between layers.[14] Several global features have been unveiled: explosive synchronization in multilayer networks,[15] synchronization driven by energy transport in interconnected networks,[16] intra-layer,[17] and cluster[18] synchronization in multiplex networks, breathing synchronization in time delayed multiplexes,[19] and global synchronization on interconnected layers as in smart grids[12] or in a *network of networks* configuration.[20]

In this work, we consider multiplex networks, i.e., the case where layers are made of a fixed set of nodes and connections exist between each node of a layer and all its replicas in the other layers, and show that a genuinely distinctive form of synchronization emerges, namely, *inter-layer synchronization*, occurring when each unit in each layer is synchronized with all its replicas, regardless of whether or not it is synchronized with the other members of its layer.

Our results are organized as follows: (i) we analytically derive the conditions for the existence and stability of such a solution, (ii) we numerically verify the analytic predictions in several cases where inter-layer synchronization emerges with or without intra-layer synchronous behaviors, (iii) we inspect the robustness of the new solution against a progressive de-multiplexing of the structure, and (iv) we give experimental evidence of inter-layer synchronization with nonlinear electronic circuits.

II. MODEL AND METHODS

We start by considering two layers of identical structure, formed by N identical m dimensional dynamical systems whose states are represented by the vectors $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ (top layer) and $\mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N\}$ (bottom layer) with $\mathbf{x}_i, \mathbf{y}_i, \in \mathbb{R}^m$ for $i = 1, 2, \ldots, N$, as depicted in Fig. 1. As already mentioned, the inter-layer synchronous state $\mathbf{X} \equiv \mathbf{Y}$[21] can be realized with or without intra-layer synchronization. The former case (Fig. 1 left) corresponds to all nodes in both layers following the same trajectory, and it therefore reduces to the classical scenario of a globally synchronous solution whose stability can be accounted for by the Master Stability Function (MSF).[1,22] The latter case (Fig. 1 right), instead, is far more general, as it only requires that every node i in each layer be synchronous to its replica in the other layer $[\mathbf{x}_i(t) = \mathbf{y}_i(t), \forall i]$, with unconstrained intra-layer dynamics.

Let the dynamics (in the absence of inter-layer coupling) be

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_j \mathcal{L}_{ij}\, \mathbf{h}(\mathbf{x}_j),$$

and

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) - \sigma \sum_j \mathcal{L}_{ij}\, \mathbf{h}(\mathbf{y}_j),$$

where $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ are the autonomous evolution and output vectorial functions, σ is the intra-layer coupling strength, and $\mathcal{L}_{ij}$ are the elements of the Laplacian matrix encoding the intra-layer topology. In this setting, the layer's dynamical state will be, in general, different at all times, i.e., $\mathbf{X}(t) \neq \mathbf{Y}(t)$. Let us now consider the multiplex structure

$$\begin{aligned}\dot{\mathbf{X}} &= \mathbf{f}(\mathbf{X}) - \sigma\mathcal{L} \otimes \mathbf{h}(\mathbf{X}) + \lambda\,[\mathbf{H}(\mathbf{Y}) - \mathbf{H}(\mathbf{X})],\\ \dot{\mathbf{Y}} &= \mathbf{f}(\mathbf{Y}) - \sigma\mathcal{L} \otimes \mathbf{h}(\mathbf{Y}) + \lambda\,[\mathbf{H}(\mathbf{X}) - \mathbf{H}(\mathbf{Y})],\end{aligned} \quad (1)$$

where the inter-layer coupling is realized through the output vectorial function $\mathbf{H} : \mathbb{R}^m \to \mathbb{R}^m$ and the inter-layer coupling strength is λ. Notice that, if the coupling between layers is diffusive, the inter-layer synchronous state *always* exists, and the manifold $\mathbf{X}(t) = \mathbf{Y}(t)$ is an invariant set whatever value the coupling constants may take.

Let now $\delta\mathbf{X}(t) = \mathbf{Y}(t) - \mathbf{X}(t)$ be the vector describing the difference between the dynamics of the two layers. Considering a small $\delta\mathbf{X}$ and expanding around the inter-layer synchronous solution $\mathbf{Y} = \mathbf{X} + \delta\mathbf{X}$ up to first order, one obtains a set of $N \times m$ linearized equations for the perturbations $\delta\mathbf{x}_i$

$$\delta\dot{\mathbf{x}}_i = [J\mathbf{f}(\tilde{\mathbf{x}}_i) - 2\lambda J\mathbf{H}(\tilde{\mathbf{x}}_i)]\delta\mathbf{x}_i - \sigma \sum_j \mathcal{L}_{ij} J\mathbf{h}(\tilde{\mathbf{x}}_j)\, \delta\mathbf{x}_j, \quad (2)$$

where J denotes the Jacobian operator and $\tilde{\mathbf{X}} = \{\tilde{\mathbf{x}}_i\}$ is the state of one isolated layer obeying

$$\dot{\tilde{x}}_i = \mathbf{f}(\tilde{\mathbf{x}}_i) - \sigma \sum_j \mathcal{L}_{ij}\, \mathbf{h}(\tilde{\mathbf{x}}_j). \quad (3)$$

The linear equations (2), solved in parallel to the $N \times m$ nonlinear equations (3) for $\dot{\tilde{x}}_i$, allow calculating all Lyapunov exponents transverse to the manifold $\mathbf{X} = \mathbf{Y}$. The maximum of those exponents (MLE) as a function of the parameter pair (σ, λ) actually gives the necessary conditions for the stability of the inter-layer synchronous solution: whenever MLE < 0, perturbations transverse to the manifold die out, and the multiplex network is said to be *inter-layer synchronizable*.

In the following, the intra- and inter-layer synchronization errors, respectively, defined as:

$$E_{intra} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{j\neq 1} \|\mathbf{x}_j(t) - \mathbf{x}_1(t)\| dt, \quad (4)$$

$$E_{inter} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\delta\mathbf{X}(t)\| dt, \quad (5)$$

and the MLE are calculated by performing numerical simulations of Eqs. (1) and (2), respectively ($\|\|$ stands for the Euclidean norm in Eqs. (4) and (5)). Without lack of generality, we consider two possible kinds of topologies where both layers are either (i) Erdös-Renyi[23] (ER) or (ii) scale-free[24] (SF), in all cases with $N = 500$ Rössler oscillators,[25] whose autonomous evolution is given by $\mathbf{f}(\mathbf{x}) = [-y - z, x + 0.2y, 0.2 + z(x - 9.0)]$. ER and SF networks are generated by means of the procedures proposed in Refs. 23 and 24, respectively, and therefore the considered SF networks display a degree distribution $p(k) \propto k^{-3}$.

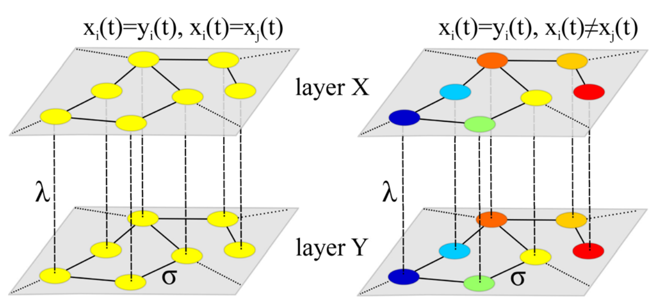

FIG. 1. Schematic representation of a multiplex of two layers of identical oscillators, and of the two types of inter-layer synchronization: with (left) and without (right) intra-layer synchronization. Labels σ and λ denote the intra- and inter-layer coupling strengths, respectively. Each node i (j) in the top (bottom) layer is an m dimensional dynamical system whose state is represented by the vector $\mathbf{x}_i$ ($\mathbf{y}_j$).

III. THEORETICAL RESULTS

We start by setting $\mathbf{h} = (0, 0, z)$ so that the corresponding MSF is in class I, thus preventing the occurrence of intra-layer synchronization for any possible value of σ at $\lambda = 0$. In addition, the inter-layer coupling function is taken to be $\mathbf{H} = (0, y, 0)$, which generates a class II MSF at $\sigma = 0$. According to the classification introduced in Ref. 1, dynamical systems can be classified as: (i) class I, when their MSF is always positive, (ii) class II, when their MSF is always negative for values higher than a given normalized coupling threshold ν_1, and (iii) class III, when their MSF is negative inside an interval bounded by ν_1 and ν_2, which are in that case the two zeros of the MSF. This classification can be adapted to layers, just by considering each of them as a unique dynamical system with $N \times m$ dimensions. Indeed, one can then speak of a class I, II, or III layer according to the synchronization properties previously defined in the framework of dynamical systems.

Results are reported in Fig. 2, where E_{inter} is plotted versus λ for several values of σ, both for SF (a) and ER (b) topologies. In all cases, a smooth transition from an incoherent multiplex dynamics with $E_{inter} > 0$ to an inter-layer synchronous evolution where $E_{inter} = 0$ is observed, always in the absence of intra-layer synchronization [insets in Figs. 2(a) and 2(b) show that E_{intra} remains well above zero for the whole explored range of λ]. In Fig. 2(c), the MLE for the SF case is plotted, showing that E_{inter} vanishes exactly at the same λ at which the MLE gets negative, thus confirming the validity of the analytical approach. To gather a clearer view on the impact of the network heterogeneity, Fig. 2(d) reports the critical coupling λ^* (the value of λ at the onset of inter-layer synchronization) as a function of σ, for both SF and ER topologies, and several average degrees. As in single layer networks, multiplexes of heterogeneous structures require smaller coupling thresholds to sustain a stable synchronous state. There is a non-monotonic relationship between the synchronization threshold and the *stiffness* within each layer (as measured by σ). The horizontal (vertical) dashed line in Fig. 2(d) [Figs. 2(a) and 2(b)] indicates the threshold λ^* for the appearance of a synchronous state at $\sigma = 0$, obtained by analyzing a pair of bidirectionally coupled Rössler systems. More rigid layers need larger inter-layer couplings to synchronize (as one would expect), but beyond a certain point in the rigidity, the trend is remarkably reversed.

While keeping $\mathbf{H} = (0, y, 0)$, a much richer scenario occurs in the case $\mathbf{h} = (x, 0, 0)$, where the uncoupled layers ($\lambda = 0$) are of class III (intra-layer synchronization is stable within an interval), and therefore, inter- and intra-layer synchronization can, in principle, coexist. We make use of an ER multiplex network of $\langle k \rangle = 16$ to show the interplay of both types of synchronization.[26] The results are reported in Fig. 3. In particular, the left panel shows that the predictions of the MSF still guarantee an error on E_{intra} of the order of 1% (see the white contour line). This shows how intra-layer synchronization is only mildly affected by the presence of inter-layer couplings. The middle and right panels of Fig. 3 report E_{inter} and the MLE, respectively. In both panels, the vertical dashed line further marks the synchronization transition point predicted by the MSF for two coupled oscillators ($\sigma = 0$). Three different regions (A, B, and C) can be identified in the parameter space: inter-layer without (region A) and with (region B) intra-layer synchronization, and an area (region C) where intra-layer synchronization occurs without inter-layer synchronization. In the right panel, the isoline (white thick curve) marks the points where the MLE changes its sign from positive to negative, and shows that, at intermediate values of σ, inter-layer synchronization is realized for values of λ below the synchronization threshold of a

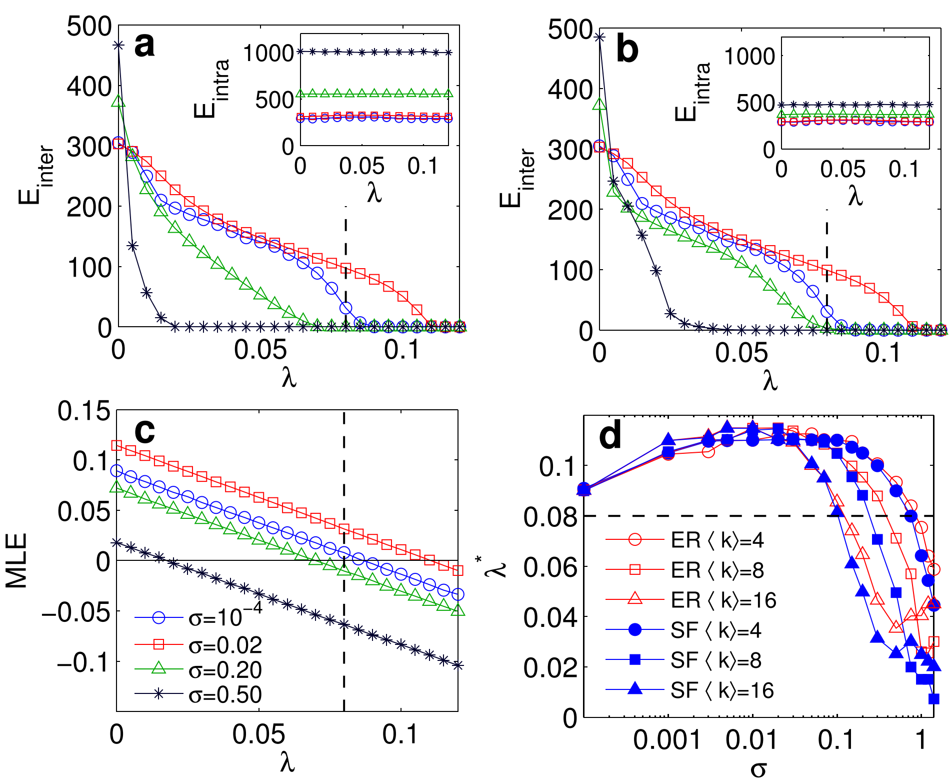

FIG. 2. Inter-layer synchronization for class-I layers. (a) E_{inter} (see main text for definition) in multiplexes of SF layers of $N = 500$ Rössler oscillators and (c) the corresponding MLE as a function of λ for several intra-layer couplings σ (see legend in panel (c)). (b) The same as in (a) but for multiplexes of ER layers. Insets in (a) and (b) show E_{intra} in the top layer, and the vertical dashed line is the synchronization coupling threshold for a pair of nodes ($\sigma = 0$) coupled through the y variable. Each point is an average of 10 realizations, with $\langle k \rangle = 8$. (d) Dependence of the inter-layer synchronization onset, λ^*, on the intra-layer coupling σ for ER (red hollow symbols) and SF (blue solid symbols) layers, and different mean degrees. The horizontal dashed line is placed at the same value as the vertical line in panels (a)–(c).

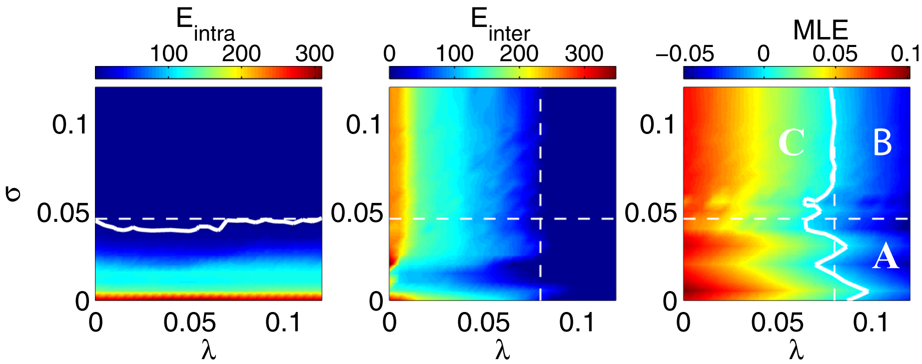

FIG. 3. Intra- and inter-layer synchronization for class-III layers. The intra- (left) and inter-layer (middle) synchronization errors (see main text for definitions), and the MLE (right) in the (σ, λ) parameter space. Color codes are shown in the upper bars. In all panels, the horizontal dashed lines mark the synchronization threshold of each isolated layer ($\lambda = 0$). In the middle and right panels, the vertical dashed line marks the synchronization threshold of a pair of nodes ($\sigma = 0$). The white contour line in the left (right) panel is the isoline corresponding to $E_{intra} = 0.01E_{intra}^{max}$ (the isoline where the MLE changes its sign from positive to negative). See main text for the description of regions A, B, and C in the right panel. Each point is an average over 10 multiplexes realizations with $\langle k \rangle = 16$.

pair of oscillators (vertical dashed line). A second remarkable conclusion is that, in a multiplexed structure, inter- and intra-layer synchronization may enhance each other.

Further insight can be gathered by exploring the robustness of the inter-layer synchronous state under a progressive de-multiplexing of the structure. In particular, the study is performed starting from a complete multiplex, and with the coupling scheme $\mathbf{h} = (0, 0, z)$, $\mathbf{H} = (0, y, 0)$ as in Fig. 2. For both the SF and ER architectures and starting from the complete multiplex, we then sequentially remove the links between nodes and their corresponding replicas, until the two layers become uncoupled. In Fig. 4, E_{inter} is reported as a function of the actual number of multiplexed nodes, from N to 0, with a disconnecting mechanism following either a random sequence or the increasing/decreasing degree ranking. Robustness is critically dependent on the balance between the inter- and intra-layer couplings. At relatively low and balanced couplings (left panel), E_{inter} grows as soon as the first pair of replica nodes is disconnected, and almost at the same rate regardless on the node sequence. A radically different situation occurs when the intra-layer coupling considerably exceeds the inter-layer one (right panel): inter-layer synchronization persists even if a large fraction of nodes are de-multiplexed. Furthermore, multiplexes with homogeneous structured layers (void symbols) are less robust than those formed by SF layers (solid symbols), and engineering a multiplex with synchronous layers is actually tantamount to coupling just a fraction of the largest degree nodes in each layer. This behavior holds even when the hubs of the SF multiplex are sequentially disconnected [see squares of Fig. 4(b)]. Notice indeed that, in analogy with what reported for network's targeting,[14] only 25 (110) of the largest degree nodes maintain $E_{inter} = 0$ in SF (ER) multiplexes of size $N = 500$.

IV. EXPERIMENTAL VALIDATION

Finally, we report experimental evidence of inter-layer synchronization in nonlinear electronic circuits, with the setup sketched in Fig. 5(a). The experiment consists of an electronic array, a personal computer (PC), 14 analog to digital converters (ADCs), and 4 digital outputs (DOs) ports from a multifunctional data acquisition (DAQ) card controlled by Labview. The ADCs are used for sampling one of the state variable out of all the networked circuits, and the DOs are used as controllers for the gain of the two coupling strengths σ and λ. The array is made of 14 Rössler-like circuits arranged in two identical layers (blue nodes), each one of them having two different electronic couplers, one for the coupling among nodes in the same layer (σ) and the second for the interaction of each node with its replica in the other layer (λ).

The chaotic dynamics of the circuits is well approximated by

$$\frac{dx_i}{dt} = -\alpha_1 \left[(x_i + \beta y_i + \Gamma z_i) - \left(\sigma\psi \sum_{j=1}^{N} a_{ij}[x_j - x_i] \right) \right], \tag{6a}$$

$$\frac{dy_i}{dt} = -\alpha_2(-\gamma x_i + (1-\delta) y_i), \tag{6b}$$

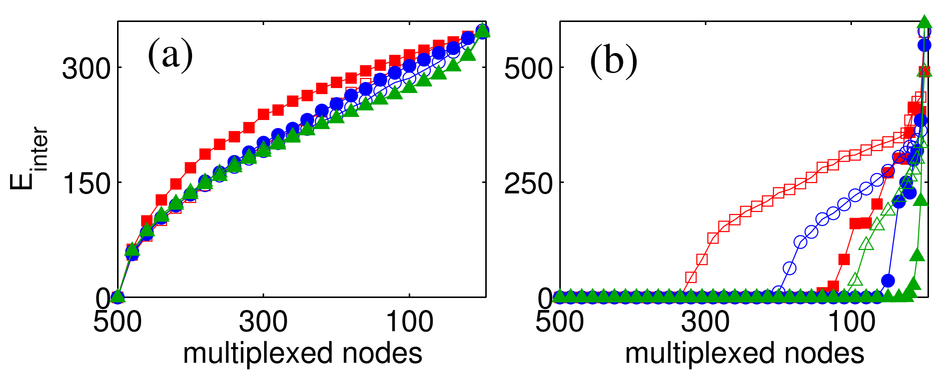

FIG. 4. Robustness of the inter-layer synchronization. E_{inter} vs. the number of multiplexed nodes for ER (void symbols) and SF (solid symbols) configurations. From a full multiplex, nodes are progressively disconnected following a random (blue circles), and a decreasing (red squares) or increasing degree (teal triangles) sequence. Inter-layer coupling is $\lambda = 0.1$ and intra-layer coupling values are (a) $\sigma = 0.1$, and (b) $\sigma = 1.0$. Points are averages over 20 network realizations, with $\langle k \rangle = 8$.

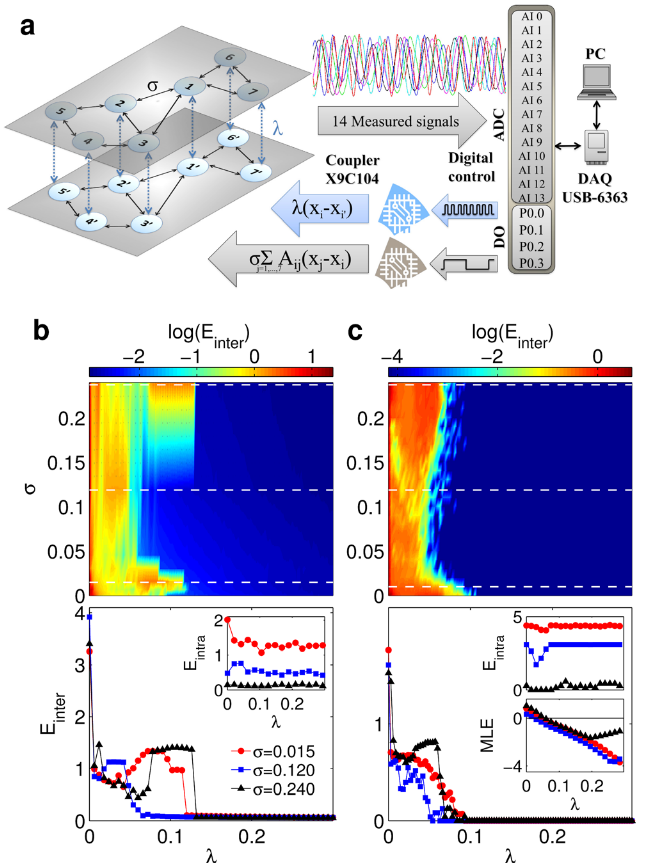

FIG. 5. Experimental evidence of inter-layer synchronization. (a) Experimental setup. The left image is a sketch of the coupling topology of the 14 electronic circuits composing the multiplex network (see main text for the description of the experimental procedure used). The whole experiment is controlled from a PC with Labview Software. (b) and (c) Color maps of E_{inter} (log scale) in the parameter space (σ, λ) (top panels) and for three specific σ values (bottom panels, color codes in the legend) calculated experimentally (b) and via numerical simulations (c). Insets show the corresponding values of E_{intra}. In panel (c), the MLE is also reported as a separate inset.

$$\frac{dz_i}{dt} = -\alpha_3(-G_{x_i} + z_i), \tag{6c}$$

$$G_{x_i} = \begin{cases} 0, & \text{if } x_i \leq 3 \\ \mu(x-3), & \text{if } x_i > 3, \end{cases} \tag{6d}$$

where σ is the tunable intra-layer coupling strength and the rest of parameter values are gathered in Table I. The reader is referred to Ref. 27 for a detailed description of the experimental implementation of the Rössler like circuit in networks, and Refs. 20 and 28 for previous realizations in different network configurations. Both the intra- and inter-layer couplings are realized through the x variable.

The coupling is adjusted using two digital potentiometers X9C104, whose parameters $C_{u/d}$ (up/down resistance) and C_{step} (increment of the resistance at each step) are controlled by digital signals coming from a DAQ Card, P0.0-P0.3. The outputs of the circuit are sent to a set of voltage followers that act as a buffer, and then to the analog ports (AI0-AI13) of the same DAQ Card. At each σ value (starting from $\sigma = 0$), λ is initially set to zero, and then the polarization voltage of the circuits is turned off and on, after a waiting time of 500 ms. The signals corresponding to the x state variables of the 14 circuits are acquired by the analog ports AI0-AI13 and saved in the PC for further analysis. λ is then incremented by one step, and the procedure is repeated 100 times (until the maximum value of λ is reached). When the entire run is finished, σ is increased by one step, and another cycle of λ values is initiated.

TABLE I. Parameter values of the chaotic dynamics of one Rössler like circuit as described in Eq. (6).

$\alpha_1 = 106.3830$	$\alpha_2 = 42.5532$	$\alpha_3 = 2127.7$
$\beta = 10$	$\Gamma = 20$	$\gamma = 50$
$\delta = 10$	$\mu = 15$	$\psi = 20$

The experimental (Fig. 5(b)) and numerical (Fig. 5(c)) results for E_{inter} and E_{intra} are in very good agreement for the entire parameter space (σ, λ), indicating that our analytical predictions actually apply also for slightly nonidentical systems, as the electronic circuits contain resistors and capacitors of 1% and 10% tolerance, respectively, causing a small deviation with respect to the synchronization region predicted by the MSF approach.[28]

V. CONCLUSION

In conclusion, we provided a full characterization of inter-layer synchronization, a novel and distinctive dynamical phenomenon occurring in multiplex networks of identical layers, in terms of its stability conditions, its relation to intra-layer synchronization and network topology, and its robustness under partial de-multiplexing of the network. We further reproduced it experimentally for slightly nonidentical systems, indicating that the phenomenon is robust enough to be observable in the presence of noise and parameter mismatch. Our results, therefore, suggest the way of unveiling the new dynamics in a variety of multiplexed real world systems.

Our MSF approach strictly requires that the synchronization manifold exists as an invariant solution. In its turn, this implies the requirement that the coupled layers in the multiplex need to be identical. Therefore, we were forced to explicitly require identical structures of connectivity in the two layers as the very essential condition for developing our treatment, a framework where the Laplacians of the two layers are not identical proving so far analytically intractable. Previous experience in the study of synchronization in simple (i.e., one-layer) networks indicates that predictions based on the MSF approach turn out to be valid (to very good approximation) also when the coupled systems differ moderately. Evidently, a rigorous approach should be abandoned in such a case, and one should rely on some kind of (reasonable) approximations. Future work in this direction is underway in order to extend the scope of our research to more realistic scenarios.

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